Objectives: Introduction to Steiner Trees.

When finding a minimal spanning tree, we choose edges with minimum weight that already exist in a graph. We’ll be doing something similar here, but extending our search to edges that aren’t specified beforehand. In the context of Steiner trees, our weights are actual distances, and we will allow ourselves to add new edges to the graph, if that would be advantageous.

Here’s an example to illustrate the basic idea. Consider the following, very simple graph. We have three vertices and three edges in the shape of an equilateral triangle. Each pair of vertices is a distance $\sqrt{3} \approx 1.732$ miles apart.

These three vertices could represent three observation towers that need to be wired together with communication cables. In search for a shortest possible network, we could consider a minimal spanning tree. In this case, any two edges form a minimal spanning tree, and one choice is shown below on the left.

The length of the minimal spanning tree is

$$\sqrt{3} + \sqrt{3} \approx 1.732 + 1.732 = 3.464.$$ 

Is there a shorter network? If we don’t restrict ourselves to the sides of the triangle, the answer is yes.

If we add an extra vertex at the center of the triangle, it turns out that the distance from the center to each of the vertices is 1 mile. Using the three edges from the center vertex gives us a tree with length

$$1 + 1 + 1 = 3,$$

and this is indeed shorter.
**Example.** Here's another example. With four points, which are the vertices of a 2 by 4 rectangle, one tree connecting the points is shown on the left. It has total length
\[
2 + 4 + 2 = 8.
\]

Adding two vertices, we can get the tree on the right, which also connects the four vertices. It’s length is approximately
\[
1.15 + 1.15 + 2.85 + 1.15 + 1.15 = 7.45,
\]
which is shorter.

**Steiner Points.** The best possible extra vertices to add will always have three edges, and these edges will always make three $120^\circ$ angles. These best possible new vertices are called Steiner points. The corresponding shortest networks are called Steiner trees. If it is possible to shorten a minimal spanning tree in a graph by adding vertices and edges, then the shortest possible network will always be a Steiner tree.

**Shortest tree for a square.** You’re going to be finding a Steiner tree for a square. To help you out, here are some basic facts about triangles. In the figure below, the relationships between the sides of a 30-60-90 triangle and a 45-45-90 triangle are shown.

The one on the left is known as a 30-60-90 triangle, and the figure is basically saying that the longest side (the hypotenuse) is twice as long as the shortest side, and the other side is $\sqrt{3}$ times as long as the shortest side.

The triangle on the right is called a 45-45-90 triangle. The two shorter sides are the same length, and the long side is $\sqrt{2}$ times as long as either of the shorter sides.
We're going to find the lengths of the networks shown below. The one on the right has two Steiner points (i.e., they make 120° angles).

1. First of all, we have four vertices that are the corners of a 2 × 2 square. A minimal spanning tree is shown on the left in the picture above. Find the total length of this minimal spanning tree.

The middle figure shows a tree formed by adding a vertex in the middle.

2. Copy the middle figure on your paper, and draw in the four sides of the 2 × 2 square.

3. In your picture, you should now have four 45-45-90 triangles, and they're all exactly the same shape and size. What is the length of the longest side for each triangle?

4. Based on what I've told you about 45-45-90 triangles, you should be able to figure out the lengths of the shorter sides. What is that length?

5. Find the total length of the middle network.

6. Is this better than the minimal spanning tree on the left?

OK. Now we're going to look at the Steiner tree on the right. Those angles around the Steiner points are 120° angles.

7. Copy the figure on the right on your paper, and draw in the sides of the square again.

8. The two triangles on the right and left of your picture are exactly the same shape and size. What is the length of their longest side?

9. OK. Now extend that middle horizontal edge to cut our two triangles in half. You should now have four 30-60-90 triangles, all the same size and shape. What is the length of the medium-length side on each triangle?

10. From what I told you about 30-60-90 triangles, find the length of the short sides.

11. Now the long sides.

12. You should now be able to find the length of the horizontal middle edge in the original Steiner tree.

13. Find the total length of the Steiner tree.

14. Which of the three networks is the shortest?

Answers on last page.
Facts about Steiner trees

Here are some facts about Steiner trees that you will use in answering some of the Practice and Quiz Problems.

120° angles. Steiner points will always make three 120° angles. You can use this fact to find where your Steiner points should be. You could, for example, have a sheet of clear plastic with three lines coming together at 120° angles, and given a graph, you can slide this sheet around on top of the graph, until the three lines hit three vertices on your graph.

Steiner trees or minimal spanning trees. Given a graph, the shortest possible network will either be a minimal spanning tree or a Steiner tree. In some cases, it will be impossible to fit a Steiner point into a graph, or even if you can, the best Steiner tree might be longer than a minimal spanning tree. In particular, adding a point other than a Steiner point will never give you the shortest possible network.

For example, in the first set of Practice Problems, adding the middle point to the square gave you a network shorter than the minimal spanning tree. This tells you two things. First, the minimal spanning tree was not the shortest network, and therefore, there must be a shorter network that is a Steiner tree. Second, the middle point was not a Steiner point, and therefore, the middle network of length 5.656 couldn’t possibly be the shortest possible total length.

Multiple Steiner trees. There may be more than one possible Steiner tree, and these may be of different lengths. One of these Steiner trees or the minimal spanning tree must be the shortest possible network.

Practice Problems

15. Suppose you have a graph, and a minimal spanning tree has total length 20. By adding a point that is not a Steiner point, you can construct a network with length 19. What do you know about the length of the shortest possible network?

16. Suppose a minimal spanning tree has length 20, there is only one way to construct a Steiner tree, and that Steiner tree has length 21. What do you know about the length of the shortest network?

17. A minimal spanning tree has length 20, there are only two ways to construct a Steiner tree. One Steiner tree has length 19 and one 18. What do you know about the length of the shortest network?

18. A minimal spanning tree has length 20, there are at least two ways to construct a Steiner tree, and one of the Steiner trees has length 19. What do you know about the length of the shortest network?

Answers on next page.
1) $2 + 2 + 2 = 6.$
3) 2.
4) The longest side is $\sqrt{2}$ times longer than the short sides, so the short sides must have length $\frac{2}{\sqrt{2}} \approx 1.414$.
5) $1.414 + 1.414 + 1.414 + 1.414 = 5.656$ (approximately)
6) Yes.
8) 2.
9) 1.
10) $\frac{1}{\sqrt{3}} \approx 0.577$.
11) $0.577 \cdot 2 = 1.154$ (approximately)
12) That middle edge plus two short sides equals 2, so $2 - 0.577 - 0.577 = 0.846$.
13) $1.154 + 1.154 + 0.846 + 1.154 + 1.154 = 5.462$ (approx.)
14) The Steiner tree is the shortest.
15) The shortest possible network must have length shorter than 19.
16) The shortest network has length 20.
17) The shortest network has length 18.
18) The shortest network has length 19 or shorter.