OK. I want to look at finding an explicit formula for the Fibonacci sequence. We’ve seen some simple examples, like

\[ 3, 6, 12, 24, 48, \ldots \]

where each term is 2 times the term before. In mathematical terminology, we might say that this sequence satisfied the \textit{relational equation}

\[ A_n = 2 \cdot A_{n-1}. \]

Also important is that the first term is \( A_0 = 3 \). In a simple case like this, we can figure out that the explicit formula is

\[ A_n = 3 \cdot 2^n. \]

It turns out that if we have a simple relation like this, an explicit formula of the form

\[ A_n = C x^n \]

will describe any sequence that satisfies a relational equation like the one above.

\textbf{Example.} Consider the sequence

\[ 5, 15, 45, 135, \ldots , \]

which satisfies the relations

\[ A_n = 3 \cdot A_{n-1}, \text{ and } A_0 = 5. \]

With a relation this simple, you can pretty much see the explicit formula, but I want to work through a process that will work in more complicated cases.

\textit{Step 1.} Guess that the explicit formula will take the form

\[ A_n = C x^n. \]

\textit{Step 2.} Plug this guess into the relational equation. Since \( A_{n-1} = C x^{n-1} \), we substitute for \( A_n \) and \( A_{n-1} \) to get

\[ C x^n = 3 \cdot C x^{n-1}. \]

\textit{Step 3.} Solve for \( x \). In this case, we can divide both sides by \( C \) and \( x^{n-1} \). This leaves us with

\[ x = 3. \]

Of course, we knew that already. Anyway, we now know that the explicit formula is

\[ A_n = C \cdot 3^n \]

for some number \( C \).
Step 4. Solve for $C$, but using the value of $A_0$, which we know two ways. First, we’re given that
\[ A_0 = 5, \]
and second, we have the explicit formula
\[ A_0 = C \cdot 3^0. \]
Having $A_0$ two ways gives us the equation
\[ 5 = C \cdot 3^0 \]
\[ 5 = C \cdot 1 \]
\[ 5 = C. \]
This makes the explicit formula
\[ A_n = 5 \cdot 3^n, \]
which is what we expected.

Example. OK. Here’s an example similar to the ones we’ve done, but the explicit formula isn’t quite as obvious. The sequence is
\[ 4, 6, 36, 54, 324, 486, 2916, 4374 \ldots \]
Specifically, the pattern is that each term is nine times the term that is two places before it. Do you see that? The relational equation is
\[ A_n = 9 \cdot A_{n-2}. \]
We want to find an explicit formula for this sequence. The trick is to guess that the explicit formula takes the form
\[ A_n = C \cdot x^n. \]
So just like before, we want to plug this guess into the relational equation. Note that $A_{n-2} = C \cdot x^{n-2}$.
Therefore, we get
\[ C \cdot x^n = 9 \cdot C \cdot x^{n-2}. \]
Now, for $n = 10$, this last equation would be
\[ C \cdot x^{10} = 9 \cdot C \cdot x^8, \]
and if we divided both sides by $C$ and $x^8$, we would get
\[ x^2 = 9. \]
Similarly, for $n = 15$, we would have
\[ C \cdot x^{15} = 9 \cdot C \cdot x^{13}, \]
and dividing by $C$ and $x^{13}$ would give us
\[ x^2 = 9. \]
In general, if we divide $C \cdot x^n = 9 \cdot C \cdot x^{n-2}$ by $C$ and $x^{n-2}$, we get
\[ x^2 = 9. \]
This equation tells us that $x$ must equal 3 or $-3$. Going back to our guess for $A_n$, we see that
\[ A_n = C \cdot (3)^n \quad \text{or} \quad A_n = C \cdot (-3)^n. \]
are possible explicit formulas. It turns out that these aren’t the only possible explicit formulas, but we’re close. It may not be entirely obvious, but it turns out that the sum of any two explicit formulas also satisfies the same relational equation, and all the possible explicit formulas take the form
\[ A_n = B \cdot 3^n + C \cdot (-3)^n. \]
All we have to do is to find the constants $B$ and $C$ that go with our sequence that starts with $A_0 = 4$ and $A_1 = 6$. We do this by setting up equations. We know $A_0$ two different ways. It’s equal to 4, and it’s also
\[ A_0 = B \cdot 3^0 + C \cdot (-3)^0 = B + C. \]
Therefore, we know
\[ B + C = 4. \]
We also know $A_1$ two different ways, $A_1 = 6$ and
\[ A_1 = B \cdot 3^1 + C \cdot (-3)^1 = 3B - 3C, \]
and so we have another equation
\[ 3B - 3C = 6. \]
The two equations together form a system of equations
\[ \begin{align*}
B + C &= 4 \\
3B - 3C &= 6
\end{align*} \]
One way of solving a system of equations like this is to multiply the equations by numbers and adding the equations together. In this case, we could multiply the first equation by 3 to get
\[ \begin{align*}
3B + 3C &= 12 \\
3B - 3C &= 6
\end{align*} \]
Adding the two equations together gives you
\[ 6B - 0C = 18, \]
or
\[ B = 3. \]
Plugging this into the equation $B + C = 4$, we see that
\[ C = 1. \]
Therefore, our explicit formula must be
\[ A_n = 3 \cdot 3^n + 1 \cdot (-3)^n. \]
As a quick check,
\[ A_2 = 3 \cdot 3^2 + 1 \cdot (-3)^2 = 3 \cdot 9 + 1 \cdot 9 = 27 + 9 = 36. \]

**QUIZ 28 WORKSHEET**

OK. Now it’s your turn. Consider the sequence
\[ 3, 2, 12, 8, 48, 32, 192, \ldots \]
The first term is $A_0 = 3$, the second term is $A_1 = 2$, and the pattern is that each term is four times the term two before it. In other words,
\[ A_n = 4 \cdot A_{n-2}. \]
Step 1. We guess that the explicit formula will take the form
\[ A_n = C \cdot x^n. \]
1. You’ll need to plug in $A_{n-2}$. So $A_{n-2} = C \cdot x^{??}$. What is the missing exponent? Enter the expression with no spaces.

In Step 2, we plug our guess into the relational equation $A_n = 4 \cdot A_{n-2}$. Do that. You should get
\[ Cx^2 = ?? \cdot Cx^{??}. \]
2. What goes in the single question mark?
3. What goes in the double question mark?
4. What goes in the triple question mark?

   In Step 3, we solve for $x$, by cancelling the $C$’s, and as many $x$’s as we can. Do that. You should end up with $x^2 = ??$.

5. What goes in the single question mark?
6. What goes in the second question mark?
7. There are two solutions to the equation. The positive solution is $x = ?$.

8. The negative solution is $x = ?$.

   In Step 4, since we have two possible solutions, we know that the explicit formula we’re looking for takes the form

   \[ A_n = B \cdot 2^n + C \cdot (-2)^n. \]

   (45)

   To solve for both $B$ and $C$, we’ll need to use $A_0 = 3$ and $A_1 = 2$. According to our possible explicit formulas (involving $B$ and $C$), we have

   \[ A_0 = B \cdot 2^0 + C \cdot (-2)^0 = B + C \]  
   \[ (46) \]

   \[ A_1 = B \cdot 2^1 + C \cdot (-2)^1 = 2B - 2C \]  
   \[ (47) \]

   Since we need to have $A_0 = 3$ and $A_1 = 2$, we have two equations.

9. The first equation is $B + C = ?$.

10. The second equation is $2B - 2C = ?$

    If we divide the second equation by $2$, we get the system

    \[ B + C = 3 \]  
    \[ B - C = 1 \]  

11. (3 points) Add the two equation together, and then solve for $B$. What is $B$ equal to?

12. (3 points) You can plug your answer from Problem 11 into either of equations in the system to find $C$. What is $C$ equal to?

13. (4 points) Which of the following is our explicit formula? Just enter the letter like $b$

    a. $A_n = 3 \cdot 2^n + 2 \cdot (-2)^n$.
    b. $A_n = 1 \cdot 2^n + 2 \cdot (-2)^n$.
    c. $A_n = 2 \cdot 2^n + 1 \cdot (-2)^n$.
    d. $A_n = 3 \cdot 2^n + 1 \cdot (-2)^n$. 