MA 1128: Lecture 06 – 9/19/16

Graphs
Functions
When an equation has more than one variable, it will almost always have an infinite number of solutions.

For example, the equation

\[ x^2 + y^2 = 4 \]

has infinitely many pairs of \( x \)'s and \( y \)'s that satisfy the equation. The values \( x = 2 \) and \( y = 0 \), for example, give us

\[ (2)^2 + (0)^2 = 4 + 0 = 4. \]

The values \( x = 1 \) and \( y = \sqrt{3} \) give us \( (1)^2 + (\sqrt{3})^2 = 1 + 3 = 4. \)

Both of these pairs of values for \( x \) and \( y \) are called solutions.
Example (cont.)

We will always think of $x$ as being the *first* variable, and $y$ the second. We’ll use the *ordered pair* $(2,0)$ to mean “$x = 2$ and $y = 0$”.

We'll use the ordered pair $(1, \sqrt{3})$ for $x = 1$ and $y = \sqrt{3}$.

In other words, both of these ordered pairs are solutions to the equation

$$x^2 + y^2 = 4.$$
Example (cont.)

Still looking at the equation $x^2 + y^2 = 4$, if we were to consider the ordered pair $(1.50, 1.32)$, and we tried it in the equation, we would get

$$(1.50)^2 + (1.32)^2 = 2.25 + 1.7424 = 3.9924,$$

which is almost 4, but not quite. This is not a solution, but we might say that it is an *approximate solution*.

On the other hand, the ordered pair $(3, 1)$ gives us $(3)^2 + (1)^2 = 9 + 1 = 10$, which is not even close to 4.

This is *not* a solution. How should we talk about which ordered pairs are solutions, and which are not?

Since this equation with two variables has an infinite number of solutions, we couldn’t possibly list them all out.

If we wanted to describe the solution set exactly, the best we could do is say something like, “all of the solutions to $x^2 + y^2 = 4$,” which isn’t very useful.
A common, and more useful, description uses a *graph*. We’ll do this with two number lines for the variables $x$ and $y$. The $x$-axis is almost always drawn horizontally, and the $y$-axis vertically. In this picture, we have the graph of the four points just mentioned. For example, the point $(1, \sqrt{3})$ is above $x = 1$ and across from $y = \sqrt{3} = 1.732…$
Practice Problems

1. Look at the four points drawn in the graph on the previous screen. List them out in order from highest (up) to lowest (down). (The lowest one is (2,0).)
2. List them out in order from furthest right to furthest left.

Click for answers.
1) Highest – (1, \sqrt{3}), (1.50,1.32), (3,1), (2,0) – Lowest.
2) Right – (3,1), (2,0), (1.50,1.32), (1,\sqrt{3}) – Left.
Graph of an Equation

The *graph of an equation* is the graph of all the solutions to the equation drawn together.

The graph of an equation is generally a straight line or a smooth curve.

The equation that we’ve been looking at, \( x^2 + y^2 = 4 \), has a graph that is a circle.

This graph and the four points we’ve been talking about are in the picture on the next screen.

Note that the two solutions, (2,0) and (1, \( \sqrt{3} \)), lie on the graph.

The approximate solution, (1.50, 1.32) looks like it lies on the graph, but is a little off the circle to the inside.

The point (3,1), which is not a solution, lies well off the graph.
Example (cont.)

Note that the two solutions, \((2,0)\) and \((1, \sqrt{3})\), lie on the graph. The approximate solution, \((1.50, 1.32)\) looks like it lies on the graph, but is just to the inside. The point \((3,1)\), which is not a solution, lies well off the graph.
Example (cont.)

Every point on the circle is a solution.
For example, there is a point on the circle above \( x = -0.8 \).
In the picture below, we follow the line straight up from \( x = -0.8 \) until we hit the graph. This is the point corresponding to this solution.
To get the \( y \)-coordinate, we look across to the \( y \)-axis.
It looks like, maybe, \( y = 1.81 \). (the actual value is closer to 1.83, but we’re just eye-balling it.)
To check, \( (-0.8)^2 + (1.81)^2 = .64 + 3.2761 = 3.9161 \), which is fairly close to 4.
We would have to look very closely at the graph to get a more accurate solution.
Practice Problems

One approximate solution to \( x^2 + y^2 = 4 \) is \((-0.8, 1.81)\).

1. Find another approximate solution that has \( x \)-coordinate \( x = -0.8 \).
2. Find two approximate solutions that have \( x \)-coordinate \( x = 1.2 \) (Look closely at the graph, and do the best you can.)
3. Find two approximate solutions that have \( y \)-coordinate \( y = -1.5 \)

Click for answers:
1) \((-0.8, -1.81)\) is on the bottom part of the circle. Anything close to \((-0.8, -1.83)\) is fine.
2) \((1.2, \pm 1.6)\) are exact solutions, so anything close to these is fine.
3) \((\pm 1.3, -1.5)\) is pretty close. Anything around 1.3 is fine.
Drawing Graphs

You need to be able to read graphs, like you’ve just done, and also to be able to sketch the graph of an equation.

One method that works pretty well goes like this.

1. Graph a bunch of solutions,
2. Draw the graph through these points, as best you can.

**Basic Principle:** If you’ve got enough solutions, then the graph will be the simplest and smoothest curve that passes through all of the points.

There are exceptions to this, of course, but they’re fairly obvious, and we’ll talk about these as we come across them.
Example

Consider the equation $y = 3x - 2$.

This equation is convenient for graphing, since it has the $y$ by itself on one side. We can find solutions by picking $x$’s and finding the corresponding $y$-values.

The $x$-value $x = 0$ is always easy. This gives $y = 3(0) - 2 = -2$. Therefore, $(0, -2)$ must be a solution.

It doesn’t matter too much what we choose, so we’ll just pick a few more.

If $x = 3$, then $y = 3(3) - 2 = 9 - 2 = 7$. So $(3, 7)$ is a solution.

If $x = -2$, then $y = 3(-2) - 2 = -6 - 2 = -8$. So $(-2, -8)$ is a solution.

If $x = 1$, then $y = 3(1) - 2 = 3 - 2 = 1$. So $(1, 1)$ is a solution.
Example (cont.)

We’ve found four points on the graph of this equation, (0,-2), (3,7), (-2,-8), and (1,1).
They are plotted below on the left.
On the right, the simplest curve that fits all of the points is a straight line.
Practice Problems

1. For the equation $y = 2x + 1$, find solutions corresponding to $x = -2, 0, 1, 3$. Plot these points, and draw the simplest curve through them (should be a straight line).

2. For the equation $y = -x + 2$, find four solutions (you choose the $x$’s). Plot your points and draw the graph.

Click for answers

1) $(-2, -3), (0, 1), (1, 3), (3, 7)$

2)
More on Drawing Graphs

What I’ll generally do when I plot points to draw a graph is the following.
1. Make a table with a column for my $x$’s and a column for my $y$’s.
2. Choose some values for $x$.
3. Find the corresponding values for $y$.
4. Plot the points.
5. Draw the curve (or line).
Example: The equation $y = x^2$

Make table

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<tr>
<td></td>
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<td></td>
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<td></td>
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<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Choose x’s

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Find y’s

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Plot solutions

Draw the curve

Next Slide
Practice Problems

1. For the equation $y = x^2$, I have the bottom part of the graph pretty rounded. Carefully plot points corresponding to $x = -1, -.6, -.4, -0.2, 0, .2, .4, .6,$ and $1$, and graph the curve to see what the shape should be. Two places past the decimal point should be sufficient.

   Answers: $(-1.0, 1.00), (-0.6, 0.36), (-0.4, 0.16), (-0.2, 0.04), (0.0, 0.00), (0.2, 0.04), (0.4, 0.16), (0.6, 0.36),$ and $(1.0, 1.00)$. 
Functions

When we were plotting points to graph an equation, we used equations that looked like $y = 3x - 2$ or $y = 2x^2 + 3$. We also looked at the graph of the equation $x^2 + y^2 = 4$. If you try to find a couple of solutions to $x^2 + y^2 = 4$, you’ll find that this equation is not as convenient to use as an equation like $y = 2x^2 + 3$ is. For this and many other reasons, mathematicians have discovered that it’s a very good idea to work with equations like $y = 2x^2 + 3$ as much as possible.

When $y$ is by itself on one side, and there are only $x$’s on the other, we will say that we’re writing $y$ as a function of $x$.

We can also have $x$ as a function of $y$, but we won’t consider this very often.
Practice Problems

1. Rewrite the equation $3x + 2y = 6$, so that $y$ is a function of $x$.
2. Rewrite $3x + 2y = 6$, so that $x$ is a function of $y$.

Click for answers:

1) $y = 3 - \frac{3}{2} x$; 2) $x = 2 - \frac{2}{3} y$
Function Notation

Mathematicians have found that a special function notation is very useful.

Whenever we have $y$ as a function of $x$, as in $y = 2x^2 + 3$, we will replace the $y$ by $f(x)$ (read “$f$ of $x$”).

$$f(x) = 2x^2 + 3$$

Be careful, “$f(x)$” is **not** multiplication.
Think of “$f(x)$” and “$y$” as being equivalent.
Example

This *function notation* is used mainly as follows.

If \( f(x) = 2x^2 + 3, \)
then \( f(a) = 2a^2 + 3, \)
and \( f(2) = 2(2)^2 + 3, \)
and \( f(-4) = 2(-4)^2 + 3. \)

Whatever we put in the parentheses gets substituted for the \( x \)'s on the other side.
Example

If \( f(x) = x^2 - 2x + 7 \),
Then \( f(2) = (2)^2 - 2(2) + 7 \)
\[ = 4 - 4 + 7 \]
\[ = 7. \]
And also \( f(-3) = (-3)^2 - 2(-3) + 7 \)
\[ = 9 + 6 + 7 \]
\[ = 22. \]

Note that it’s a very good idea to substitute into parentheses. The second one, \( f(-3) \), might get especially confusing otherwise.
Practice Problems

Let \( f(x) = 3x - 1 \).

1. Find \( f(2) \).
2. Find \( f(-1) \).

Let \( f(x) = -2x^2 + x + 1 \).

3. Find \( f(4) \).
4. Find \( f(-2) \).

Click for answers:
1) \( f(2) = 5 \); 2) \( f(-1) = -4 \); 3) \( f(4) = -27 \); 4) \( f(-2) = -9 \).