MA 1128: Lecture 15 – 4/6/2018

Adding and Subtracting
Rational Expressions (cont.)
Common Multiples

At the end of Lecture 14, we were adding and subtracting rational expressions. Doing this is pretty straightforward, *as long as the denominators are the same.*

Let’s look at regular fractions again before we continue with rational expressions. Suppose we want to add $\frac{1}{2}$ and $\frac{1}{3}$. We need to change these so that the denominators are the same. The new denominator needs to be a multiple of 2, and also a multiple of 3. In other words, we need a *common multiple* of 2 and 3. The smallest common multiple is 6, but 12, 18, 24, etc. will work also.
Example (cont.)

We can multiply $\frac{1}{2}$ by $\frac{3}{3}$ to get $\frac{3}{6}$.

$\frac{1}{2}$ and $\frac{3}{6}$ are equal, since all we did was multiply by $1$.

For $\frac{1}{3}$, we use $\frac{1}{3}$ times $\frac{2}{2}$ to get $\frac{2}{6}$.

We will generally write this as follows.

\[
\frac{1}{2} + \frac{1}{3} = \frac{1}{2} \cdot \frac{3}{3} + \frac{1}{3} \cdot \frac{2}{2} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}
\]

A common multiple is $12$ in this next example.

\[
\frac{7}{4} + \frac{2}{3} = \frac{7}{4} \cdot \frac{3}{3} + \frac{2}{3} \cdot \frac{4}{4} = \frac{21}{12} + \frac{8}{12} = \frac{29}{12}
\]

We’ll leave our answer as $\frac{29}{12}$. Mathematicians never use mixed numbers.
Practice Problems

Add or subtract these fractions.
1. $\frac{2}{3} + \frac{1}{4}$.
2. $\frac{3}{5} - \frac{1}{2}$.
3. $\frac{1}{6} + \frac{1}{5}$.

Answers:
1) $\frac{11}{12}$
2) $\frac{1}{10}$
3) $\frac{11}{30}$
Least Common Denominators

Any common multiple in the denominator (a.k.a., common denominator) will work, but smaller numbers are usually easier to work with. This becomes much more important with rational expressions. Therefore, we’ll want the least common denominator.

For example, consider $\frac{1}{27} + \frac{1}{18}$.

In the previous examples, the common denominator was just the product of the original denominators.

We can do that here, of course. $27 \cdot 18 = 486$.

\[
\frac{1}{27} + \frac{1}{18} = \frac{1}{27} \cdot \frac{18}{18} + \frac{1}{18} \cdot \frac{27}{27} = \frac{18}{486} + \frac{27}{486} = \frac{45}{486}
\]

This simplifies.

\[
\frac{45}{486} = \frac{9 \cdot 5}{9 \cdot 54} = \frac{5}{54}
\]
Note that the 54 is a common multiple of 27 and 18.
We could (and should) have used 54 instead of 486.

There is a systematic way of finding the least common divisor/multiple.
It looks at the prime factors.
Since $27 = 3 \cdot 3 \cdot 3$, the least common denominator needs to have at least three 3’s.
The other denominator factors as $18 = 2 \cdot 3 \cdot 3$, so we also need a 2 (we already have three 3’s, which is more than enough).
The least common denominator is therefore, $3 \cdot 3 \cdot 3 \cdot 2 = 54$
Example

Consider \( \frac{1}{36} - \frac{1}{54} \).

First we need the least common denominator.

\[
36 = 2 \cdot 18 = 2 \cdot 2 \cdot 9 = 2 \cdot 2 \cdot 3 \cdot 3.
\]

The least common denominator needs at least two 2’s and two 3’s.

\[
54 = 2 \cdot 27 = 2 \cdot 3 \cdot 9 = 2 \cdot 3 \cdot 3 \cdot 3.
\]

The 2 is already covered, and we already have two 3’s. We need one more 3.

The least common denominator must be \( 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 108 \).

\[
\frac{1}{36} - \frac{1}{54} = \frac{1}{36} \cdot \frac{3}{3} - \frac{1}{54} \cdot \frac{2}{2} = \frac{3}{108} - \frac{2}{108} = \frac{1}{108}
\]
Practice Problems

1. Find the least common multiple of 300 and 90.
2. Compute $7/300 + 7/90$.
3. Find the least common multiple of 270 and 63.

Answers:
1) 900
2) 91/900
3) 1890
4) 23/1890
Rational Expressions

Consider the sum \( \frac{1}{x^2 + 2x + 1} + \frac{1}{x^2 - 1} \).

We need a common denominator, and better yet, a least common denominator.
Just like with numbers, we’ll factor the denominators.

\( x^2 + 2x + 1 = (x + 1)(x + 1) \), so the least common denominator has to have these two factors.

The other denominator is a difference of squares, so \( x^2 - 1 = (x + 1)(x - 1) \).
The \( x + 1 \) is already covered, but we also need an \( x - 1 \).
The least common denominator is \( (x + 1)(x + 1)(x - 1) \).

\[
\frac{1}{x^2 + 2x + 1} + \frac{1}{x^2 - 1} = \frac{1}{(x + 1)(x + 1)} + \frac{1}{(x + 1)(x - 1)} = \frac{1}{(x + 1)(x + 1)} \cdot \frac{(x - 1)}{(x - 1)} + \frac{1}{(x + 1)(x - 1)} \cdot \frac{1}{(x + 1)}
\]

\[
= \frac{(x - 1)}{(x + 1)(x + 1)(x - 1)} + \frac{(x + 1)}{(x + 1)(x + 1)(x - 1)} = \frac{(x - 1) + (x + 1)}{(x + 1)(x + 1)(x - 1)} = \frac{2x}{(x + 1)(x + 1)(x - 1)}
\]
Practice Problems

Compute the following. (In your answers, multiply the numerator out and simplify. Leave the denominator factored. This is how the choices in the quiz will be written.)

1. \( \frac{1}{x^2 + 2x + 1} + \frac{1}{x^2 - x - 2} \).

2. \( \frac{1}{x^2 - x - 2} + \frac{1}{x^2 - 3x + 2} \).

Answers:

1) \( \frac{2x-1}{[(x+1)(x+1)(x-2)]} \)

2) \( \frac{2x}{[(x+1)(x-2)(x-1)]} \)
Example

Here’s one more example, \( \frac{x - 3}{x^2 - 1} - \frac{2}{x^2 - x - 2} \).

In problems like these, I would like your answer to have the numerator multiplied out and simplified and your denominator left factored. If you can factor the numerator, go ahead and see if anything will cancel.

\[
\begin{align*}
\frac{x - 3}{x^2 - 1} - \frac{2}{x^2 - x - 2} &= \frac{x - 3}{(x + 1)(x - 1)} - \frac{2}{(x + 1)(x - 2)} \\
&= \frac{(x - 3)(x - 2)}{(x + 1)(x - 1)(x - 2)} - \frac{2(x - 1)}{(x + 1)(x - 1)(x - 2)} \\
&= \frac{(x - 3)(x - 2) - 2(x - 1)}{(x + 1)(x - 1)(x - 2)} = \frac{x^2 - 5x + 6 - (2x - 2)}{(x + 1)(x - 1)(x - 2)} \\
&= \frac{x^2 - 7x + 8}{(x + 1)(x - 1)(x - 2)}
\end{align*}
\]
Practice Problems

Find the least common denominator, add or subtract, and simplify like the last example.

1. \(\frac{x - 2}{x^2 - x - 6} + \frac{x^2 + 2x - 1}{x^2 + 3x + 2}\).
2. \(\frac{x^2}{x^2 + 5x - 14} - \frac{x + 2}{x^2 + 6x - 7}\).
3. \(\frac{x - 2}{x^2 - 2x - 3} + \frac{3}{x^2 - x - 2}\).

Answers:
1) \(\frac{x^3 - 8x + 1}{(x + 2)(x - 3)(x + 1)}\)
2) \(\frac{x^3 - 2x^2 + 4}{(x - 2)(x + 7)(x - 1)}\)
3) \(\frac{x^2 - x - 5}{(x + 1)(x - 3)(x - 2)}\)