Rational Equations
Roots and Radicals
Equations with Rational Expressions (Example)

Especially with the addition of rational expressions, we need a common denominator.

Since we will be working with *equations*, we have the option of multiplying both sides by the same thing.

When solving equations involving rational expressions, most people like multiplying both sides of the equation by the least common denominator (LCD).

In this case, \(x^2 + 7x + 12 = (x+3)(x+4)\) is the LCD.

\[
\frac{3}{x+3} + \frac{5}{x+4} = \frac{12x+9}{x^2 + 7x + 12}
\]

\[
\frac{3}{x+3} \cdot (x+3)(x+4) + \frac{5}{x+4} \cdot (x+3)(x+4) = \frac{12x+9}{x^2 + 7x + 12} \cdot (x+3)(x+4)
\]
Example (cont.)

\[
\frac{3}{x+3} + \frac{5}{x+4} = \frac{12x+9}{x^2 + 7x + 12}
\]

\[
\frac{3}{x+3} \cdot (x+3)(x+4) + \frac{5}{x+4} \cdot (x+3)(x+4) = \frac{12x+9}{x^2 + 7x + 12} \cdot (x+3)(x+4)
\]

\[
3(x+4) + 5(x+3) = 12x + 9
\]

\[
3x + 12 + 5x + 15 = 12x + 9
\]

\[
-4x = -18
\]

\[
x = \frac{9}{2}
\]

One thing that you should check for in a problem like this is to make sure the solutions are valid. Here, \(x = -3\) and \(x = -4\) would make the denominators zero in the original equation. This would be bad, so if either of these came up as a solution, we would throw them out.
More Examples

In the first example below, the least common denominator is 10.
In the other example, the least common denominator is \((x + 1)(x - 2)\).

\[
\frac{x}{5} = \frac{x-3}{2} \\
x \cdot 10 = \frac{x-3}{2} \cdot 10 \\
x \cdot 2 = (x-3) \cdot 5 \\
2x = 5x - 15 \\
-3x = -15 \\
x = 5
\]

\[
\frac{2}{x+1} = \frac{1}{x-2} \\
\frac{2}{x+1} \cdot (x+1)(x-2) = \frac{1}{x-2} \cdot (x+1)(x-2) \\
2 \cdot (x-2) = 1 \cdot (x+1) \\
2x - 4 = x + 1 \\
x = 5
\]
Practice Problems

Find the solution to each of the following equations.

1. \[ \frac{3}{8} + \frac{3}{4} = \frac{x}{5}. \]
2. \[ \frac{x + 1}{x + 10} = \frac{x - 2}{x + 4}. \]
3. \[ \frac{4x - 1}{x^2 + 5x - 14} = \frac{1}{x - 2} - \frac{2}{x + 7}. \]

Answers:
1) \( x = \frac{45}{8} \)
2) \( x = 8 \)
3) \( x = \frac{12}{5} \)
Radicals and Roots

We talked about exponents awhile ago.
Remember that we use them to indicate repeated multiplication.
For example, \(3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81\).
It is convenient to have a notation for the inverse operation.
In this case, if we have \(3^4 = 81\), then we’ll say “the fourth root of 81 is 3.”
In mathematical symbols, (the little “house” is called a radical symbol)
\[
\sqrt[4]{81} = 3
\]
Since \((-3)^4 = (-3)(-3)(-3)(-3) = 81\), -3 is also a fourth root of 81.
To avoid any confusion, we’ll make it clear which fourth root we want.
\[
\sqrt[4]{81} = 3
\]
\[
-\sqrt[4]{81} = -3
\]
Roots and Radicals (cont.)

Since $4^2 = 16$, we’ll say that $4$ is a second root of $16$.
Second roots are so common, they have a special name, and are called square roots. We also don’t write the little 2.

$$\sqrt[2]{16} = \sqrt{16} = 4$$
$$-\sqrt{16} = -4$$

A third root is usually called a cube root.

Example, what is $\sqrt[3]{8}$?

What cubed is 8? Well, $2 \cdot 2 \cdot 2 = 8$, so $\sqrt[3]{8} = 2$
Practice Problems

Give the positive root only.

1. $\sqrt{4}$
2. $\sqrt[3]{27}$
3. $\sqrt{49}$
4. $\sqrt[4]{16}$

Answers:
1) 2; 2) 3; 3) 7; 4) 2.
Rational (or Fractional) Exponents

It turns out that we can write roots with exponents instead of radicals. They maybe don’t look as nice, but they’re much easier to work with. Essentially, a second root is equivalent to an exponent of $\frac{1}{2}$.

A third root is equivalent to an exponent of $\frac{1}{3}$.

A fourth root is equivalent to an exponent of $\frac{1}{4}$.

For example,

$$\sqrt[3]{x} = x^{\frac{1}{3}}$$

$$\sqrt[5]{7} = 7^{\frac{1}{5}}$$

All of the usual rules for exponents apply.
With Calculators

This notation works with our calculators really well, and it gives us options on how we use the calculator.

Your calculator should have a square root button.

To compute the square root of 16, you would enter 16, square-root button, and get the answer 4. (On some calculators, you hit the square-root button first.)

You should also have an exponent button on your calculator (it probably has $y^x$, $x^y$, or ^ on it). [[That last exponent below is a .5]]

\[
\sqrt{16} = \sqrt[2]{16} = 16^{\frac{1}{2}} = 16^{.5}
\]

Enter 16, exponent button, .5, =, and the answer should be 4.

Since you may not have a fourth-root button on your calculator, you can

\[
\sqrt[4]{16} = 16^{\frac{1}{4}} = 16^{.25}
\]

Enter 16, exponent button, .25, =, answer is 2.
Practice Problems

ROUND TO 4 DECIMAL PLACES. You’ll need to be very careful to round correctly. For example, 1.732050808 would round to 1.7321, since a “5” in the fifth decimal place rounds up.

1. \(\sqrt[4]{625}\)
2. \(\sqrt[5]{10}\)
3. \(\sqrt[4]{2}\)

Answers:
1) 5; 2) 1.5849; 3) 1.1892
More on Rational Exponents

You’ll often see regular exponents mixed with exponents from roots. Just remember that the regular exponents go on top, and the roots go on the bottom.

\[ 3\sqrt{x^5} = x^{\frac{5}{3}} = \left(3\sqrt{x}\right)^5 \]

\[ 4\sqrt{7^3} = 7^{\frac{3}{4}} = 7^{.75} = 4.3035 \quad \text{Rounded to 4 decimal places.} \]
Practice Problems

Round to 4 decimal places.

1. \( \sqrt[5]{5^3} \)

2. \( \sqrt{3^5} \)

Answers:
1) 2.6265
2) 15.5885
Multiplying and Dividing with Radicals

Note that radicals (roots) are essentially the same as exponents, so they “distribute” over multiplication and division. (BUT NOT ADDITION AND SUBTRACTION!!)

For example,

\[
\sqrt[2]{4} = \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5}
\]

\[
\sqrt[3]{2} = \frac{\sqrt[3]{1}}{\sqrt[3]{8}} = \frac{1}{2}
\]

\[
\sqrt{4x^2} = \sqrt{4} \sqrt{x^2} = 2x
\]
Practice Problems

Simplify as much as you can.

1. \( \sqrt{\frac{9}{16}} \)
2. \( \sqrt[4]{9x^4} \)
3. \( \sqrt[3]{\frac{4}{32}} \)

Answers:
1) \( \frac{3}{4} \); 2) \( 3x^2 \); 3) \( \frac{1}{2} \).