MA 1128: Lecture 19 – 4/20/2018

Quadratic Formula
Solving Equations with Graphs
Completing-the-Square Formula

One thing you may have noticed when you were completing the square was that you followed the same steps every time.

1. Move the constant to the other side.
2. Take half of the $x$-coefficient, square it, and add that to both sides.
3. Factor the perfect square trinomial.
4. Take the square root of both sides (plus/minus).
5. Move the constant to the other side.

Done.

A procedure like this can usually be gathered up into a formula. The question is, is it easier to remember and use the procedure? Or is it easier to remember and use the formula? In this case, the formula’s easier for most people. Actually, factoring is easier, but only if the equation is really nice.
Before we look at the formula, let’s extend the completing-the-square procedure to the general quadratic equation.

For example, if we had the quadratic equation $3x^2 + 9x - 15 = 0$, the procedure from last time wouldn’t work because the $x^2$-coefficient is 3 (and not 1).

This is easy to fix, however. Just divide both sides by 3 (note 0 divided by 3 is 0). Now we have $x^2 + 3x - 5 = 0$, and we know what to do from here.
Practice Problems

For both of these problems, divide by the $x^2$-coefficient and complete the square. Stop when you get to the form $(x + A)^2 = B$.

1. $5x^2 + 20x - 15 = 0$.
2. $3x^2 + 15x - 3 = 0$.

Answers:
1. $(x + 2)^2 = 7$
2. $(x + 5/2)^2 = 29/4$
The Quadratic Formula

Any quadratic equation can be written in the form

\[ ax^2 + bx + c = 0. \]

If you go through the completing-the-square procedure on this equation (divide by \( a \), complete the square, etc.), you will end up with the following formula.

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

This is the quadratic formula,

and it gives you the solutions in terms of the coefficients \( a, b, \) and \( c \).
Example

Consider the quadratic equation $x^2 - 7x + 12 = 0$.
Here the $x^2$-coefficient is $a = 1$.
The $x$-coefficient is $b = -7$.
And the constant term is $c = 12$.
We get the solutions from the quadratic formula by filling in the values for $a, b,$ and $c$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(12)}}{2(1)}$$

These are the solutions. One comes from the $+$ in the $\pm$,
and the other comes from the $-$.
Example (cont.)

Of course, we’ll always want to simplify as much as we can.

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(12)}}{2(1)}
\]

\[
= \frac{7 \pm \sqrt{49 - 48}}{2} = \frac{7 \pm \sqrt{1}}{2} = \frac{7 \pm 1}{2}
\]

\[
= \frac{8}{2}, \frac{6}{2} = 4, 3
\]

In this case, factoring would have been easier. In other cases, factoring might not be.

\[(x - 3)(x - 4) = 0, \text{ and } x = 3, 4.\]
Example

Factoring would be a bit harder on $5x^2 - 2x - 3 = 0$, but the quadratic formula is about the same difficulty as with the other example.

The coefficients are $a = 5$, $b = -2$, and $c = -3$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(5)(-3)}}{2(5)}$$

$$= \frac{2 \pm \sqrt{4 + 60}}{10} = \frac{2 \pm \sqrt{64}}{10} = \frac{2 \pm 8}{10}$$

$$= \frac{10}{10}, \frac{-6}{10} = 1, \frac{-3}{5}$$
Example

Consider the quadratic equation $3x^2 + 7x + 1 = 0$.
Here, $a = 3$, $b = 7$, and $c = 1$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-7 \pm \sqrt{(7)^2 - 4(3)(1)}}{2(3)}$$

$$= \frac{-7 \pm \sqrt{49 - 12}}{6} = \frac{-7 \pm \sqrt{37}}{6}$$

Since 37 doesn’t have a simple square root, we’ll just leave the exact answer like this.

To get approximate solutions, just plug the numbers into your calculator.

The steps for the $+$-solution are:
7, +/- (to change to $-7$), $+$, 37, square root, $=$ (to finish off the numerator), $\div$, 6, $=$.

You should get $x = -0.152872912$. The other solution is $x = -2.180460422$.
Practice Problems

Express your answers in decimal form rounded (correctly) to 4 decimal places. You should keep all of the digits you get from your calculator until the end.

1. \(2x^2 + 4x - 6 = 0\).
2. \(5x^2 - 2x - 7 = 0\).
3. \(2x^2 - 10x + 3 = 0\).
4. \(4x^2 - 2x = 6\). (Get 0 on the right side first!)

Answers:
1. 1, -3;
2. 1.4, -1;
3. 4.6794, 0.3206;
4. 1.5, -1
Other Kinds of Equations

We can always solve linear equations \((ax + b = 0)\)
and quadratic equations \((ax^2 + bx + c = 0)\).
Cubics (like \(ax^3 + bx^2 + cx + d = 0\)) and quartics (like \(ax^4 + bx^3 + cx^2 + dx + e = 0\)) are doable, but are usually really really hard.
Beyond these, it’s even worse.
With computers, we can find approximate solutions for most equations, and this is generally sufficient for applications.
Here, we’ll look at one way using graphs.
Since we know quadratic equations pretty well, let’s use a quadratic equation to demonstrate the technique.
Consider \(x^2 – 5x + 6 = 0\).
This factors, \((x – 2)(x – 3) = 0\), so we know the solutions are \(x = 2, 3\).
Solutions from Graphs

Look at the graph of the quadratic function \( y = x^2 - 5x + 6 \) or \( f(x) = x^2 - 5x + 6 \). Note that the \( x \)-intercepts are \( x = 2 \) and \( x = 3 \). This makes sense, because the \( x \)-intercepts are the points with \( y \)-coordinate 0. When \( y = 0 \), you just have the quadratic equation \( 0 = x^2 - 5x + 6 \).
In order to solve a quadratic equation using a graph, you should graph the corresponding function, and look carefully for the $x$-intercepts.

For example, consider the quadratic equation $x^2 - 4x + 1 = 0$.

Graph the function $f(x) = x^2 - 4x + 1$. 
Example (cont.)

In this last graph, note that every fifth tick-mark is a whole number. Therefore, each tick-mark must be .2, .4, .6, .8, 1.0, 1.2, etc. There are two x-intercepts. The one on the right is between $x = 3.6$ and 3.8. If we only graph between $x = 3.6$ and 3.8, we can get a clearer view of the solution.
Example (cont.)

Now in the graph on the last screen (also shown below), each tick-mark on the $x$-axis is 0.02. Divide by five, we get 0.004, so each tick-mark is .004. Therefore, if you look really really carefully, the $x$-intercept is close to $x = 3.732$.
Example (cont.)

We could graph a smaller interval, like $x = 3.72$ to $3.74$, to get an even more accurate guess at the solution.

As it is, we can be absolutely sure that $x = 3.73$ is correct rounded to two decimal places, since this is the closest 2-digit number.
Practice Problems

The other $x$-intercept from this last example is between $x = .2$ and $.4$. The graph over this interval is shown below.
1. Give the best 3-decimal place guess you can for this $x$-intercept.
2. You should be absolutely sure of the solution rounded to 2-decimal places. What is it?

Answers:
1) $x = 0.268$;
2) $x = 0.27$
Example

Consider the equation $x^3 - 7x = -3$.
To use $x$-intercepts, we want zero on one side: $x^3 - 7x + 3 = 0$.
Then we graph $f(x) = x^3 - 7x + 3$.
Example (cont.)

There are three solutions shown in the graph (all of them, in this case). They look to be around $x = -2.8, 0.4, \text{ and } 2.4$, so I’ll graph 0.1 on either side of these.

The first one ($x = -2.9 \text{ to } -2.7$) is between $x = -2.84 \text{ and } -2.83$, and much closer to $x = -2.84$, so this is the solution to two decimal places.
Example (cont.)

The second solution is between $x = .3$ and $.5$.
It is clearly between $x = .44$ and $.45$, and much closer to $.44$.
To two decimal places, this solution is $x = .44$.
The third solution is between $x = 2.3$ and 2.5.
The new graph shows the solution to be clearly between 2.39 and 2.40. Since 2.39 is halfway between tick-marks, it’s pretty clear that the solution is closer to $x = 2.40$. 
Practice Problems

For the following equations, what function should we graph so that the x-intercepts are the solutions?
1. \(3x^4 - x^3 + x = x^2 - 11\).
2. \(x^6 - x^4 = x + x^3 - 1\).

Answers:
1) \(f(x) = 3x^4 - x^3 - x^2 + x + 11\).
2) \(f(x) = x^6 - x^4 - x^3 - x + 1\).
Practice Problems

The graph below shows two solutions. Give a .2-range (example: \( x = 2.1 \) to 2.3) that contains the indicated solution.

1. The solution on the left.
2. The solution on the right.

Answers:
1) \( x = 0.6 \) to 0.8;
2) \( x = 4.2 \) to 4.4.
Practice Problem

1. Find the solution shown below rounded correctly to 2-decimal places.

Answers:
1) $x = 4.79$
Practice Problem

1. Find the solution below rounded correctly to 2-decimal places.

Answers:
1) \( x = 0.21 \)