Functions

For the most part, the basic objects that we’ll be working with are functions. They’ll have names like $f$ or $g$ most of the time. If the function represents something specific, like area, then we might use a descriptive letter like $A$. The name could even be a word or abbreviation like sin. (Compare $f(x)$ and $\sin(x)$.)

A function will generally be associated with an input variable. Most of the time the input variable will be $x$, but in applications, we might use $t$, if $t$ represents time, for example. For trig functions, it’s traditional to use a Greek letter like $\theta$ for the input variable and assume that $\theta$ represents an angle measure. You’ll see $f(x)$, $A(t)$, and $\sin(\theta)$, and the only real difference is that different function names are being used, and different letters represent the input variable.

The main thing to remember is that the particular letters don’t really matter, but our brains seem to like consistency and using particular letters in certain situations as little clues in keeping things straight.

One simple function that we’ll talk about a lot can be described by the equation

$$f(x) = x^2.$$  

The name of the function is $f$ (although we’ll use $f$ as the name for most of our functions, so it’s not really a name). The input variable is $x$. There is also an output variable, which we’ll generally use $y$ for.

We can pretty much use $y$ and $f(x)$ interchangeably, and we might even write

$$y = f(x) = x^2.$$  

One use of the function notation is that instead of having to say “When $x$ is equal to 2, then $y$ is equal to $2^2$, which is equal to 4,”

or more symbolically

“When $x = 2$, then $y = 2^2 = 4,”$

we can combine everything into one statement

$$f(2) = (2)^2 = 4.$$  

The values for the output variable, I’ll call function values.

When you substitute a value for a variable in an expression, it’s a good idea to always enclose the thing you’re substituting in parentheses. This is especially important with negative signs.

$$f(-2) = (-2)^2 = 4.$$  

Compare this to the function $g(x) = x - x^2$. If we were interested in $x = -3$, then we would have

$$g(-3) = (-3) - (-3)^2 = -3 - 9 = -12.$$  

Quiz 01A

For these same two functions $f(x) = x^2$ and $g(x) = x - x^2$, find the following.

1. $f(0)$
2. $f(3)$
3. $f(-5)$.  

4. \( g(2) \).

5. \( g(-1) \).

**Graphs of Functions**

In calculus, we will be very interested in the exact shape of a graph. You may be familiar with the fact that the graph of the equation

\[
y = x^2
\]

is a parabola, and a parabola is \( \cup \)-shaped. That description is nowhere near sufficient for our needs. As you saw in the lab yesterday, both \( y = x^2 \) and \( y = x^{10} \) are \( \cup \)-shaped, but the two graphs do not have the same shape. For example, \( x^{10} \) is much flatter on the bottom, and the ends go to infinity much faster.

One way that we’ll address this issue is to assign numbers to how horizontal or vertical a graph is at a particular point, and another way will be that we’ll assign numbers describing how curved the graph is. One basic idea will cover both concepts.

Before we get to that, I think it’s a good idea to draw a really good graph every so often. We’ll mostly have Maple do this for us, but doing it by hand will help clear space for the graphs in our brains.

**Example 1.** Let’s draw a graph of the function \( f(x) = x^2 \). The old-fashioned way to do this is to plot some points, and then draw the nicest, simplest, and most consistent curve through them as we can. We’ll do it first with just a handful of points. Consider the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

I chose these points, because I know that the most interesting part of the graph will be around \( x = 0 \). For unfamiliar functions, I’d have to experiment a little (or maybe use some calculus). Now, we can compute the corresponding function values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Plotting these points will look something like this.
Now, we draw the nicest, simplest, and most consistent curve through these points as we can. Maybe something like this.

Quizz 01B

Do a really nice graph plotting points corresponding to \( x = -1.0, -0.9, -0.8, \ldots, 0.9, 1.0 \). That is, over the interval \(-1 \leq x \leq 1\) in steps of 0.1.

If you did a good job, you’ll notice that the middle part of the graph looks a lot like a circle. In fact, if you put a circle of radius \( r = \frac{1}{2} \) in there, it fits really well.

Here’s a Maple graph. Note that Maple does basically the same thing that you just did, but with a lot more points.
1a. Make a table containing the function values for the function $f(x) = x^3$ and the input values $x = -1.0, -0.9, -0.8, \ldots, 0.9, 1, 0$. Don’t round.

1b. Make a table containing the function values for the function $g(x) = x^7$ and the input values $x = -1.0, -0.9, -0.8, \ldots, 0.9, 1, 0$. Round correctly to three decimal places (three places to the right of the decimal point, like 14.276).

1c. Carefully plot all of these points and draw the two graphs in the same graph. You can use the grids on the next page, or use your own graph paper. Let the grid spacings correspond to a distance of 0.1 units.

2. Using the units in your graph in Problem 1, estimate the thickness of the grid lines, and the thickness of your pencil or pen lines. As a start, you might try drawing five or ten tick marks on one of the sides of one of the little squares in the grid.

3. The thicknesses of your grid lines and pencil will be greater than some of your function values, so part of your graph will be indistinguishable from the $x$-axis. At about what $x$-value will your graph of $g(x) = x^7$ start to separate from the $x$-axis?

4. How many points of the graph of $g$ actually lie on the $x$-axis?

5. We always draw graphs on paper with pencils or pens, or we use a computer to draw a graph for us on a computer screen or printed on paper. If the “real” mathematical graph consisted of the actual points, what would it look like?

Some answers: 1ab) You should have 21 points to graph for each function. For $f$, your list of points should include $(-0.7, -0.343)$ and $(0.5, 0.125)$. For $g$, your list should include $(-0.3, -0.0002187) \approx (-0.3, 0.000)$ and $(0.7, 0.0823543) \approx (0.7, 0.082)$.

1c) You’ll see some graphs in Maple on Tuesday, yours should look similar, except these will turn downwards on the left. Both graphs, and especially $g$, should pass through $(0, 0)$ very “flat” (horizontal).

2) I tried to draw ten tick marks, and they pretty much filled up the square. My pen, therefore, must draw lines about 0.01 units thick (one tenth of a grid spacing, which is 0.1). The grid lines were about the same.

3) $g(0.4) \approx 0.002$ is much smaller than the thickness of our lines, so it’s kind of hard to draw. $g(0.5) \approx 0.008$, which is about the same as the thickness of our lines, so this part of the graph and the $x$-axis might look like two lines stacked on top of each other. I would say that the graph separates from the $x$-axis somewhere around $x = 0.5$. Think about that: The graph of $g$ is basically a horizontal line segment between $-\frac{1}{2} < x < \frac{1}{2}$.

4) Just one, the point $(0, 0)$. 
5) We generally think of a point as having no width or height. A line segment or curve segment has a length, but no width. In my mind, they have to be invisible, so we can’t see a “real” graph. I’ll describe a point as being 0-dimensional, and a line or curve as being 1-dimensional. Surfaces are 2-dimensional, and I can kind of imagine seeing a 2D object, but I don’t think 2D objects exist in our universe. Do they?