Objectives: Finding the derivative, and intro to the power rule.

We have done a few computations to find the slope of a tangent line to a curve by computing the slope of nearby secant lines, and taking a limit. We have done this primarily by reasoning through numerical patterns. The advantage to a numerical pattern is that it is always generally possible to proceed that way, but away from whole numbers and nice functions, the patterns are less easy to see, and the result is not completely definitive. If possible, a more algebraic approach will lead to better results. For the easy stuff, we'll build up a catalog of formulas, and that's what we'll start doing today. Remember, however, that we tend to downplay the importance of numerical techniques, but in real-world applications, that may be all that's available.

The Binomial

Some algebraic patterns will get us started. Consider the following perfect square trinomial. The reason for using the letter \( h \) will become clear later.

\[
(x + h)^2 = x^2 + 2xh + h^2.
\]

The cube comes out as

\[
(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3.
\]

Putting these into a table establishes some patterns.

\[
\begin{array}{cccccccc}
(x + h)^0 &=& 1 \\
(x + h)^1 &=& x + h \\
(x + h)^2 &=& x^2 + 2xh + h^2 \\
(x + h)^3 &=& x^3 + 3x^2h + 3xh^2 + h^3
\end{array}
\]

Note that the powers on the \( x \)'s and \( h \)'s are the same in each row (e.g., they are all three in the last row), and all combinations of \( x \)'s and \( h \)'s are represented. Note also that the coefficients are always equal to the sum of the two coefficients directly above. Multiplying \((x + h)(x^2 + 2xh + h^2)\) will give some clues as to why that is. In any case, we can continue that pattern in the table below, which is known as Pascal’s triangle.

\[
\begin{array}{cccccccccccc}
1 & & & & & & & & & & \\
1 & 1 & & & & & & & & \\
1 & 2 & 1 & & & & & & & \\
1 & 3 & 3 & 1 & & & & & & \\
1 & 4 & 6 & 4 & 1 & & & & & \\
1 & 5 & 10 & 10 & 5 & 1 & & & & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & & & \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & & \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & \\
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 \\
1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1
\end{array}
\]

Using the patterns described, it is easy to expand the following by writing all combinations of \( x \) and \( h \) that sum to 7, and then using the coefficients from Pascal’s triangle.

\[
(x + h)^7 = x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 + 35x^3h^4 + 21x^2h^5 + 7xh^6 + h^7.
\]
1. Expand \((x + h)^6\).

**Finding the slopes of all of the tangents**

OK. So let’s get back to finding the slopes of tangent lines. Remember that when we computed a handful of slopes for the tangents to the graph of \(f(x) = x^2\), we always got double the \(x\)-value. For example, at \(x = 2\), the slope was 4. At \(x = -1\) the slope was \(-2\). In other words, at any \(x\), the slope of the tangent is \(2x\). Let’s establish this in general, and at the same time, establish our first-line procedure for finding slopes of tangents.

We will find the slope of the tangent at any particular \(x\). The point in question will have \(y\)-coordinate \(f(x)\). The point on the graph, therefore, will have coordinates \((x, f(x))\). Points on graphs will always take this form! We want a secant line to work with, so to control that other point, we’ll just pick any old number \(h\), and the other point on the secant line will have \(x\)-coordinate \(x + h\). If \(h < 0\), this other point will be to the left. If \(h > 0\), then the other point will be to the right. The \(y\)-coordinate of this other point will be \(f(x + h)\). The two points on the secant line, therefore, will be \((x, f(x))\) and \((x + h, f(x + h))\).

\[
\text{Once we have our two points, computing the slope of the secant line is the same. The two points are } (x, f(x)) \text{ and } (x + h, f(x + h)), \text{ and the slope is found by subtracting the } y\text{-coordinates, subtracting the } x\text{-coordinates, and then dividing.}
\]

\[
m = \frac{f(x + h) - f(x)}{h} = \frac{(x + h)^2 - x^2}{h}.
\]

Most of the work is in squaring \(x + h\), and that is \((x + h)^2 = x^2 + 2hx + h^2\). Plugging this in, we get

\[
= \frac{x^2 + 2hx + h^2 - x^2}{h} = \frac{2hx + h^2}{h} = 2x + h.
\]

This is a slope formula for any secant line and any \(x\) and \(h\). Letting \(h \to 0\), will give us the slope of the tangent at any point on the graph. In particular, we get

\[
\text{slope of tangent } = \lim_{h \to 0} 2x + h = 2x.
\]

**Basic Principle 1.** Given a function \(f\), the tangent line at each \(x\) has a slope. These slopes define a new function, which we will call the derivative function, and be represented with the symbol \(f'\). The function \(f'\) is called the derivative of \(f\), and it gives us the slope of the tangent line at \((x, f(x))\) on the graph for any \(x\).

We now know that given the function \(f(x) = x^2\), the derivative of \(f\) is \(f'(x) = 2x\).
We’re still looking at the function \( f(x) = x^2 \).

1. Find the coordinates of the points on the graph corresponding to \( x = -3, -1, 0, \frac{3}{2} \).

2. Find the slopes of the tangent line to the parabola at the points in problem 1.

**THE DERIVATIVE OF** \( f(x) = x^3 \).

For \( f(x) = x^3 \), the slope of a secant line would be

\[
m = \frac{f(x + h) - f(x)}{h} = \frac{(x + h)^3 - x^3}{h},
\]

and expanding \((x + h)^3 = x^3 + 3hx^2 + 3h^2x + h^3\) is similar (although a bit longer) to the expansion we did last time, so we get

\[
m = \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} = \frac{3hx^2 + 3h^2x + h^3}{h} = 3x^2 + 3hx + h^2.
\]

To get the slope of the tangent line, we take the limit as \( h \to 0 \), and find that

\[
f'(x) = 3x^2.
\]

**Extra confirmation.** Let’s do \( f(x) = x^7 \) to make sure. The slopes of the secants are

\[
m = \frac{f(x + h) - f(x)}{h} = \frac{(x + h)^7 - x^7}{h}.
\]

Expanding the \((x + h)^7\) gives

\[
m = \frac{(x + h)^7 - x^7}{h} = \frac{x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 + 35x^3h^4 + 21x^2h^5 + 7xh^6 + h^7 - x^7}{h}.
\]

The \( x^7 \)'s cancel to give

\[
m = \frac{(x + h)^7 - x^7}{h} = \frac{7x^6h + 21x^5h^2 + 35x^4h^3 + 35x^3h^4 + 21x^2h^5 + 7xh^6 + h^7}{h}.
\]

From the Pascal’s triangle stuff, we can see that the leading term will always cancel, and every term that’s left will have a factor of \( h \). Furthermore, every term will have more than one factor of \( h \), except the first. Cancelling the \( h \) in this case leaves

\[
m = \frac{(x + h)^7 - x^7}{h} = 7x^6 + 21x^5h + 35x^4h^2 + 35x^3h^3 + 21x^2h^4 + 7xh^5 + h^6.
\]

When we take the limit, all the terms go to zero, except for the first one.

\[
f'(x) = \lim_{h \to 0} \frac{(x + h)^7 - x^7}{h} = \lim_{h \to 0} 7x^6 + 21x^5h + 35x^4h^2 + 35x^3h^3 + 21x^2h^4 + 7xh^5 + h^6 = 7x^6.
\]

**HOMEWORK 06**

1. Derive the derivative formula for \( f(x) = x^6 \), like I did in the “Extra Confirmation” section.

Answers: 1) \( m = \frac{f(x + h) - f(x)}{h} = \frac{(x + h)^6 - x^6}{h} = \frac{x^6 + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6 - x^6}{h} = \frac{6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6}{h} = 6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5.

As \( h \to 0 \), the slopes go to \( 6x^5 \), since the other terms go to zero.