Objectives: Explore end behavior more precisely with asymptotes.

Tilted Asymptotes

One of the functions we looked at last time was

\[ f(x) = \frac{4x^3 - 7x^2 + 2x + 8}{2x^2 + 14x + 7} \]

The graph of \( f \) acted like \( y = 2x \), a straight line, on the ends.

A tilted asymptote is a line that the graph approaches asymptotically. In particular, the distance between the line and graph goes to zero. Clearly, in this example, \( f(x) \) does not approach \( y = 2x \) asymptotically. They tend to stay a fixed distance apart, and the actual asymptote is parallel to the line \( y = 2x \).

Asymptotic End Behavior

A lot of the time, we’re mostly interested in the general end behavior, how fast the function goes to infinity, for example, and we can see that from the dominant terms in the numerator and denominator. To get a more precise description of the end behavior, we can do the following. It’s basically applying the process of long division for integers to polynomials. We divide the denominator into the numerator as follows.

\[
\begin{array}{c|ccccc}
   & 2x & - & 35 \\
\hline
2x^2 + 14x + 7 & 4x^3 & - & 7x^2 & + & 2x & + & 8 \\
   & 4x^3 & + & 28x^2 & + & 14x \\
   \hline
   & - & 35x^2 & - & 12x & + & 8 \\
   & - & 35x^2 & - & 245x & - & \frac{245}{2} \\
   \hline
   & 233x & + & \frac{261}{2}
\end{array}
\]
The remainder hasn’t been divided by $2x^2 + 14x + 7$ yet, so we can say that

\[ f(x) = \frac{4x^3 - 7x^2 + 2x + 8}{2x^2 + 14x + 7} = 2x - \frac{35}{2} + \frac{23x + 69}{2x^2 + 14x + 7}. \]

We can’t exactly take a limit, because $f$ just goes to infinity on both ends, but as $x$ gets very large, note that the remainder part goes to zero, so the difference between $f(x)$ and $g(x) = 2x - \frac{35}{2}$ goes to zero. We’ll say that $f(x)$ approaches $g(x)$ asymptotically, and the graph of $g$ is a tilted asymptote.

**Basic Principle 1.** Given any rational function $f$, we can do a long division to find a polynomial that $f$ will approach asymptotically on the ends.

**Example 1.** Consider the rational function

\[ f(x) = \frac{4x^3 - 8x^2 + 2x + 8}{2x + 3}. \]

Doing the long division, we get

\[ \begin{array}{cccccc}
2x^2 & - & 7x & + & \frac{23}{2} \\
\hline
2x + 3 & | & 4x^3 & - & 8x^2 & + & 2x & + & 8 \\
& & 4x^3 & + & 6x^2 & & & & \\
& & & - & 14x^2 & + & 2x & + & 8 \\
& & & & - & 14x^2 & - & 21x & & \\
& & & & & & 23x & + & 8 & & \\
& & & & & & 23x & + & \frac{69}{2} & & \\
& & & & & & & \hline
& & & & & & -\frac{53}{2} & & & & \\
\end{array} \]

Therefore,

\[ f(x) = \frac{4x^3 - 8x^2 + 2x + 8}{2x + 3} = 2x^2 - 7x + 23 \frac{1}{2} - \frac{53/2}{2x + 3}, \]

and $f$ will approach the parabola $g(x) = 2x^2 - 7x + \frac{23}{2}$ asymptotically.
Example 2. We had a non-zero horizontal asymptote last time.

\[
(5) \quad f(x) = \frac{4x^3 - 7x^2 + 2x + 8}{2x^3 + 14x^2 + 7x - 103} \quad \rightarrow \quad \frac{4x^3}{2x^3} = 2. 
\]

The graph of \( f \) will approach the line \( y = 2 \) on the ends.

Applying the method from today gives us

\[
(6) \quad f(x) = \frac{4x^3 - 7x^2 + 2x + 8}{2x^3 + 14x^2 + 7x - 103} = 2 + \frac{-35x^2 - 12x + 214}{2x^3 + 14x^2 + 7x - 103},
\]

and so \( f \) approaches the constant function \( g(x) = 2 \) asymptotically. This gives us the same information as the general end behavior, and clearly, we don’t need to do long division for horizontal asymptotes.

Example 3. Similarly, we got a horizontal asymptote \( y = 0 \) in this example.

\[
(7) \quad f(x) = \frac{4x^3 - 7x^2 + 2x + 8}{2x^5 + 14x^3 + 7x^2 - 103} \quad \rightarrow \quad \frac{4x^3}{2x^5} = \frac{2}{x^2}. 
\]
Doing long division here never starts, because our function already looks like a remainder, so no need to do long division in this case either.

**Quiz 25**

Find the asymptotic end behavior for the function

\[ f(x) = \frac{2x^3 + 8x^2 + 4x + 2}{x + 3}. \]

**Homework 25**

For each of the given functions, find the polynomial that \( f \) approaches asymptotically on the ends.

1. \( f(x) = \frac{4x^3 - 4x^2 + 2x + 3}{x^2 - 3x + 8} \).
2. \( f(x) = \frac{4x^3 - 3x^2 + 2x - 4}{x + 2} \).
3. \( f(x) = \frac{12x^5 - 3x^3 + 2}{3x^8 - 3x^4 + 8x - 114} \).
4. \( f(x) = \frac{14x^4 - 7x^3 + 2x^2 - x + 1}{7x^6 - 3x^3 + 8x^2 - x + 3} \).
5. \( f(x) = \frac{2x^4 - 4x^3 + 2x^2 - 4x + 1}{x - 1} \).

Odd answers: 1) \( p(x) = 4x + 8 \).
3) \( p(x) = 0 \).
5) \( p(x) = 2x^3 - 2x^2 - 4 \).