Objectives: Continue with vector arithmetic

The standard unit vectors

It will be convenient to have names for the unit vectors that point in the positive \(x\), \(y\), and \(z\)-directions. In particular, we will define

\[
\mathbf{i} = \langle 1, 0, 0 \rangle \\
\mathbf{j} = \langle 0, 1, 0 \rangle \\
\mathbf{k} = \langle 0, 0, 1 \rangle
\]

Using our definitions for scalar multiplication and addition, we can express vectors in a third standard form.

\[
v = \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}
\]

I’ll call this last form \(ijk\)-notation, and note that it looks basically the same as the coordinate form.

Dot product

The dot product is another of our vector multiplications. [The dot product is sometimes called the scalar product, and it is a special case of something called an inner product.] While the dot product is easy to compute from the coordinate description of a vector, the definition is a little odd. In terms of the coordinates, or components, we define the dot product of \(\mathbf{u}\) and \(\mathbf{v}\) as

\[
\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1v_1 + u_2v_2 + u_3v_3.
\]

In other words, we multiply the corresponding components, which makes sense, and then we add these products together, which is the odd part. We do the same thing in any number of dimensions.

As an example, consider the following dot product

\[
\langle -4, 2, 3 \rangle \cdot \langle -2, -1, 4 \rangle = 8 - 2 + 12 = 18.
\]

In two dimensions, we might have something like

\[
\langle 3, -4 \rangle \cdot \langle 2, 5 \rangle = 6 - 20 = -14.
\]

Note that the dot product multiplies two vectors and the result is a scalar.

One important property of the dot product involves the magnitude of a vector. Consider

\[
\mathbf{v} \cdot \mathbf{v} = \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = v_1^2 + v_2^2 + v_3^2.
\]

Looking at the last expression, you should see something familiar. This is the square of the magnitude of \(\mathbf{v}\). In other words,

\[
\mathbf{v} \cdot \mathbf{v} = \| \mathbf{v} \|^2.
\]

It’s not super hard to verify, but the dot product can be interpreted geometrically with the following formula.

\[
\mathbf{u} \cdot \mathbf{v} = \| \mathbf{u} \| \| \mathbf{v} \| \cos(\theta),
\]

where \(\theta\) is the angle between \(\mathbf{u}\) and \(\mathbf{v}\). This formula gives us a relatively easy way to find the angle between two vectors given in coordinate form.

\[
\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|}
\]
Example. Find the angle between \( \mathbf{u} = \langle 1, 1 \rangle \) and \( \mathbf{v} = \langle 2, 0 \rangle \). We need the following quantities.

\[
\mathbf{u} \cdot \mathbf{v} = 2 + 0 = 2
\]

\[
\| \mathbf{u} \| = \sqrt{1 + 1} = \sqrt{2}
\]

\[
\| \mathbf{v} \| = \sqrt{4 + 0} = 2
\]

Plugging these into (12) gives

\[
\cos(\theta) = \frac{2}{\sqrt{2} \cdot 2}.
\]

Therefore,

\[
\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right).
\]

Your calculator will say that this is 45° or \( \frac{\pi}{4} \) radians. I’ll expect that you can find \( \theta \) in both degrees and radians. Don’t forget, however, that the derivatives for our trig functions assume that we’re using the radian versions of these functions, so when we are taking derivatives or anti-derivatives, use radians.

**Quiz 04A**

Consider the vectors \( \mathbf{u} = \langle 2, 1 \rangle \) and \( \mathbf{v} = \langle -1, 2 \rangle \).

1. Graph these two vectors with their tails at the origin. Do they look perpendicular?

2. If they are perpendicular, then what is the value of \( \theta \), the angle between them? I don’t care, if you do degrees or radians here.

3. What is \( \cos(\theta) \) for the \( \theta \) you said in Problem 2?

4. Compute \( \mathbf{u} \cdot \mathbf{v} \). Surprised?

**1. Cross product**

Our third kind of vector multiplication is called the *cross product*. [I think I’ve heard this also called the *vector product*.] Before defining the cross product, it kind of makes sense to define something called a *determinant* first. You’ll study the determinant in more detail in linear algebra. For now, we’ll use the determinant as a way to remember how to do the cross product, but the determinant is something that exists on its own, and it does actually have some geometric relevance here, but we won’t get into that.

OK, so there are these things called *matrices* (it’s one *matrix* and two *matrices*, by the way), which are basically rectangular arrays of numbers. Matrices with the same number of rows as columns are called *square matrices*, and the determinant is defined for square matrices. A matrix with two rows and two columns is called a \( 2 \times 2 \) matrix, and we’ll start with these. A \( 2 \times 2 \) matrix might look like

\[
\begin{bmatrix}
1 & -3 \\
2 & 6
\end{bmatrix},
\]

or in general,

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}.
\]

The determinant is then defined to be

\[
| \begin{array}{cc}
a & b \\
c & d
\end{array} | = ad - bc.
\]

Note that we’ve multiplied the diagonals, and then subtracted. Why would we define something this way? I’m not going to get into that.
The determinant of the $2 \times 2$ matrix above would be
\[
\begin{vmatrix}
1 & -3 \\
2 & 6 \\
\end{vmatrix}
= 1 \cdot 6 - (-3) \cdot 2 = 6 + 6 = 12.
\]

We will compute the determinant of a $3 \times 3$ matrix using something called a minor and the related thing called a cofactor. A $3 \times 3$ matrix might look like
\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{bmatrix}.
\]

For any entry in the matrix, the corresponding minor is the determinant of the $2 \times 2$ matrix formed by removing the row and column of that entry. For example, the 4 is in the second row and first column. Removing the second row and first column leaves the minor
\[
\begin{vmatrix}
2 & 3 \\
8 & 9 \\
\end{vmatrix}
= 18 - 24 = -6.
\]

The cofactor is the same as a minor, except if we add the row number and the column number together and get an odd number, we multiply by $-1$.

OK. It turns out that if you take any row or column, multiply the entries in that row or column by their cofactors, and then add these up, you get the determinant of the $3 \times 3$ matrix.

OK. Let’s find the determinant of the $3 \times 3$ matrix above. We’ll generally be using the first row, when we compute the cross product, so let’s use that now. The first row is 1, 2, and 3, and the multipliers for the cofactors are going to be $+1$, $-1$, and $+1$. Therefore,
\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{vmatrix}
= 1 \cdot 5 \cdot 6 + 2 \cdot (-2) \cdot 7 \cdot 9 + 3 \cdot 4 \cdot 5 \\
= 1(45-24) - 2(36-42) + 3(32-35) \\
= -3 + 12 - 9 = 0.
\]

We’d get the same thing if we used the second column, for example,
\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{vmatrix}
= -2 \cdot 4 \cdot 6 + 5 \cdot 1 \cdot 3 - 8 \cdot 1 \cdot 3 \\
= -2(36-42) + 5(9-21) - 8(6-12) \\
= 12 - 60 + 48 = 0.
\]

**Homework 04**

Write the following vectors in $ijk$ notation.

1. $\langle 2, -4, 2 \rangle$.
2. $\langle 1, 3, 0 \rangle$.
3. $\langle 4, 4 \rangle$.

Compute the dot product for the given pair of vectors.

4. $\langle 3, 3, 1 \rangle \cdot \langle 2, -1, 3 \rangle$.
5. $\langle 0, 0, 1 \rangle \cdot \langle 2, -1, 3 \rangle$.
6. $\langle 0, -5, 1 \rangle \cdot \langle 2, 6, -2 \rangle$.
7. $\langle 3, 3 \rangle \cdot \langle -1, 3 \rangle$.
8. $\langle 0, 1 \rangle \cdot \langle 5, 2 \rangle$. 
Compute the following angles.

9. Find the angle between the vectors $\mathbf{u} = \langle 1, 0 \rangle$ and $\mathbf{v} = \langle 0, 3 \rangle$. (If you know the answer, you don’t have to do any computations.)

10. Find the angle between the vectors $\mathbf{u} = \langle 1, 3 \rangle$ and $\mathbf{v} = \langle 2, 3 \rangle$.

11. Find the angle between the vectors $\mathbf{u} = \langle 1, 1, 2 \rangle$ and $\mathbf{v} = \langle 3, 5, 1 \rangle$.

HW Answers: 1) $2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$. 2) $\mathbf{i} + 3\mathbf{j}$. 3) $4\mathbf{i} + 4\mathbf{j}$.

4) $6 - 3 + 3 = 6$. 5) 3. 6) $-30 - 2 = -32$. 7) $-3 + 9 = 6$. 8) 2.

9) Since $\mathbf{u}$ points in the $x$-direction and $\mathbf{v}$ points in the $y$-direction, they must be perpendicular to each other, so $90^\circ$. Otherwise, $\cos^{-1}(0) = 90^\circ$ or $\frac{\pi}{2}$ radians.

10) $\cos^{-1}(11/\sqrt{130}) = 15.26^\circ$ or 0.266 radians.

11) $\cos^{-1}(10/\sqrt{210}) = 46.36^\circ$ or 0.809 radians.