MA 3362 Lecture 05 - Even More Examples of Rings

Friday, September 5, 2008.

Objectives: Differentiate the classes of rings with examples, continued.

A commutative ring without unity

All of the rings I’ve told you about are commutative rings with unity, even \( \mathbb{Z} \). It’s easy to find an example, however. We just talked about adding elements to a ring to make them nicer. Here, we want to get rid of the unity. Take \( \mathbb{Z} \). We don’t want 1, so toss it out. We can’t do only that, though. For example, \( 4 + (-3) \) needs an answer, since we want \( + \) to be a binary operation. There are two ways to fix this. Throw in more elements, which gets us back to where we started, or throw out the unity. Take \( \mathbb{Z} \) however. We just talked about adding elements to a ring to make them nicer. Here, we want to get rid of the unity. All of the rings I’ve told you about are commutative rings with unity, even \( \mathbb{Z} \).

A non-commutative ring

All of the rings we’ve seen so far are commutative. A standard example of this is the set of \( 2 \times 2 \) matrices with real numbers as entries and normal matrix addition and multiplication. Recall that matrix addition consists of simply adding the corresponding entries.

\[
\begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix} +
\begin{bmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{bmatrix} =
\begin{bmatrix}
  a_{11} + b_{11} & a_{12} + b_{12} \\
  a_{21} + b_{21} & a_{22} + b_{22}
\end{bmatrix},
\]

and in matrix multiplication, we multiply rows times columns in matrix multiplication. Specifically,

\[
\begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix} \begin{bmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{bmatrix} =
\begin{bmatrix}
  a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\
  a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22}
\end{bmatrix}.
\]

There is no scalar multiplication or any of the other matrix operations in this ring. In the addition, we have regular addition occurring independently in each of the four positions of the matrix. Therefore, all of the addition properties of the ring \( \mathbb{R} \) carry through. Although the multiplication is more complicated, since we only multiply one number from the first matrix times one number from the second matrix at a time, the associativity multiplication and the distributive property carry through also. Before we do that, however, let’s prove that this ring is not commutative, since that will be pretty easy. Since, commutativity must hold true for every pair of matrices, all we have to do to prove that this does not hold, is to exhibit one counter-example. Pretty much any two generic looking matrices will work. Consider

\[
\begin{bmatrix}
  1 & 2 \\
  3 & 4
\end{bmatrix} \begin{bmatrix}
  2 & 3 \\
  2 & 1
\end{bmatrix} =
\begin{bmatrix}
  1 \cdot 2 + 2 \cdot 2 & 1 \cdot 3 + 2 \cdot 1 \\
  3 \cdot 2 + 4 \cdot 2 & 3 \cdot 3 + 4 \cdot 1
\end{bmatrix} =
\begin{bmatrix}
  6 & 5 \\
  14 & 13
\end{bmatrix}.
\]

If we try to commute this multiplication, the result is

\[
\begin{bmatrix}
  2 & 3 \\
  2 & 1
\end{bmatrix} \begin{bmatrix}
  1 & 2 \\
  3 & 4
\end{bmatrix} =
\begin{bmatrix}
  2 \cdot 1 + 3 \cdot 3 & 2 \cdot 2 + 3 \cdot 4 \\
  2 \cdot 1 + 1 \cdot 3 & 2 \cdot 2 + 1 \cdot 4
\end{bmatrix} =
\begin{bmatrix}
  11 & 16 \\
  5 & 8
\end{bmatrix},
\]

which is different. This proves that we do not have \( ab = ba \) for every \( a \) and \( b \) in this set.

This is a ring, however, which we will establish. This is a bit tedious and messy, but if you just apply the definitions we have, and keep moving forward, you’ll get to the end. I’ll do some, and you can do some for homework.

We need to establish ten things. The first four concern whether the addition and multiplication are binary operations (existence and uniqueness). We also need to establish that: \( + \) is associative, \( + \) is commutative, the existence of a zero, the existence of additive inverses, \( \cdot \) is associative, and the distributive properties.
The two associative properties and the distributive property are very similar in form. Commutativity is also similar, but much simpler. The existence of a zero and additive inverses are typical of existence proofs.

**binary operations.** If you look at the definitions in equations (1) and (2), each entry is the sum of two real numbers or the products of two pairs of real numbers added together. We will assume that the basic properties of the real numbers are true and known to us, so clearly the entries of the addition and multiplication matrices are real numbers. Therefore, the sum and product of two 2×2 matrices are always 2×2 matrices, and these always exist. As for uniqueness, there is only one way a particular 2×2 matrix can be written, and so there is no possibility of the result of an addition or multiplication to come out differently. The definitions are completely explicit, so uniqueness is clear. In general, uniqueness is only an issue, if the elements of the set have non-unique representations (e.g. fractions). OK, so both + and · are binary operations.

+ is associative. I’ll do the multiplication-is-associative proof. You’ll do this one.

+ is commutative. You’ll do this one too. It’s easier than associative.

existence of a zero. You’ll do this. It’s kind of like the existence of inverses, which I’ll do.

existence of additive inverses. OK. I’m doing this one. We need to show that given any 2×2 matrix \([a_{ij}]\), there is another 2×2 matrix \([b_{ij}]\) such that

\[
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

That last matrix is the one that you should have used as the 0, by the way. OK. Let’s start the proof. We are given a matrix

\[
[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.
\]

This is a specific, but unspecified, matrix that could be any of our 2×2 matrices. Therefore, whatever we prove about this one matrix must also be true for all of them. We want to show that an additive inverse exists. With existence proofs, you typically start out with “here it is,” and then proceed to show that the thing actually works out. Just by looking at the definition, I can see what the inverse should be, and it’s

\[
[b_{ij}] = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}.
\]

That’s the start of the proof. To finish this off, I need to demonstrate that this matrix works.

\[
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} - a_{11} & a_{12} - a_{12} \\ a_{21} - a_{21} & a_{22} - a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Most algebraic proofs involve trying to prove that two things are equal, like this one. And like this one, the proof is basically a string of equalities linking one of the things to the other. Each equal sign basically marks the steps of the proof. Here, the justifications would be: (1) by the definition of matrix addition, (2) by properties of real number addition.

· is associative. Here we want to show that given any three 2×2 matrices \([a_{ij}], [b_{ij}], \text{ and } [c_{ij}]\) that

\[
([a_{ij}][b_{ij}])[c_{ij}] = [a_{ij}][(b_{ij})[c_{ij}]].
\]

OK. Let’s just dive in. We will start with the left side, and through a chain of equalities, get to the right side. We’ve got the three generic matrices, and

\[
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{22}b_{12} + a_{22}b_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},
\]
by the definition of matrix multiplication (applied to \([a_{ij}]\) and \([b_{ij}]\)). OK. Now we have two matrices to multiply together. Let’s do that next, and we get

\[
(11) = \begin{bmatrix}
(a_{11}b_{11} + a_{12}b_{21})c_{11} + (a_{11}b_{12} + a_{12}b_{22})c_{21} & (a_{11}b_{11} + a_{12}b_{21})c_{12} + (a_{11}b_{12} + a_{12}b_{22})c_{22} \\
(a_{21}b_{11} + a_{22}b_{21})c_{11} + (a_{21}b_{12} + a_{22}b_{22})c_{21} & (a_{21}b_{11} + a_{22}b_{21})c_{12} + (a_{21}b_{12} + a_{22}b_{22})c_{22}
\end{bmatrix},
\]

by the definition of matrix multiplication. Next, every entry of this matrix is a real number, so the properties of real numbers apply. In particular, we can apply the distributive property of real number algebra to get

\[
(12) = \begin{bmatrix}
a_{11}b_{11}c_{11} + a_{12}b_{21}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12} + a_{12}b_{21}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{22}c_{22} \\
a_{21}b_{11}c_{11} + a_{22}b_{21}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{22}c_{22} & a_{21}b_{11}c_{12} + a_{22}b_{21}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{22}c_{22}
\end{bmatrix},
\]

Up to this point, I have just computed the operations that I could. You should pretty much always do that. Now, I need to reverse the process and get the right side of equation (9). You could, if you wanted, actually start with the right side of equation (9), and if you get to the same matrix as expression (12), then the two sides must be equal. I prefer having just a single chain of equalities, however. You can kind of see the real number associative property at work here, and that kind of points the way. I want to get a’s factored out to the left. First, I will rearrange the terms, so the matching a’s are together.

\[
(13) = \begin{bmatrix}
a_{11}b_{11}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{21}c_{11} + a_{12}b_{22}c_{21} & a_{11}b_{11}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{21}c_{12} + a_{12}b_{22}c_{22} \\
a_{21}b_{11}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{21}c_{11} + a_{22}b_{22}c_{21} & a_{21}b_{11}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{21}c_{12} + a_{22}b_{22}c_{22}
\end{bmatrix},
\]

by the commutative property of real number addition. Now, we have

\[
(14) = \begin{bmatrix}
a_{11}(b_{11}c_{11} + b_{12}c_{21}) + a_{12}(b_{21}c_{11} + b_{22}c_{21}) & a_{11}(b_{11}c_{12} + b_{12}c_{22}) + a_{12}(b_{21}c_{12} + b_{22}c_{22}) \\
a_{21}(b_{11}c_{11} + b_{12}c_{21}) + a_{22}(b_{21}c_{11} + b_{22}c_{21}) & a_{21}(b_{11}c_{12} + b_{12}c_{22}) + a_{22}(b_{21}c_{12} + b_{22}c_{22})
\end{bmatrix},
\]

by the distributive property of real number algebra. We’re working backwards here, so this will look weird, but this is definitely true. We have

\[
(15) = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} \begin{bmatrix}
b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\
b_{21}c_{11} + b_{22}c_{21} & b_{21}c_{12} + b_{22}c_{22}
\end{bmatrix},
\]

by the definition of matrix multiplication (in reverse). Multiplication one more time in reverse, and we have

\[
(16) = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} \begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{bmatrix} \begin{bmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{bmatrix},
\]

which is what we wanted. Note again, that you could have worked from expression (16) to expression (12) 

**distributive property.** I’ll let you do this one, since this one is very similar to the last one, except not quite as messy.

**Quiz/Homework 05**

You will be filling in the missing proofs that the 2 x 2 matrices are a ring.

1. Prove that + is associative.
2. Prove that + is commutative.
3. Prove the existence of a zero.
4. Prove the distributive property (just do the \(a_{ij}(b_{ij} + c_{ij}) = a_{ij}b_{ij} + a_{ij}c_{ij}\) version - the other one is pretty much the same).
5. The ring of 2 x 2 matrices is not commutative, but it is a ring with unity. Find the unity, and prove that it is, in fact, a unity.