1. **Hilbert’s Axioms**

In [Hilbert], Hilbert has a list of 20 axioms, which were meant to transform Euclid’s postulates into a full-blown axiom system that logically lays out Euclidean geometry (what I’m calling Euclid’s geometry). What that means is that all the theorems that we would expect to be true in Euclidean geometry from these axioms. Officially, there are no pictures. There is only a list of statements.

In his book, Hilbert’s 20 axioms are organized by numbers:

- I, 1-8: Axioms of Incidence
- II, 1-4: Axioms of Order
- III, 1-5: Axioms of Congruence
- IV: Axiom of Parallels
- V, 1-2: Axioms of Continuity

We will be most concerned with just a few of them.

- I, 1. For every two points \( A, B \) there exists at least one line that contains each of the points \( A, B \).
- I, 2. For every two points \( A, B \) there exists no more than one line that contains each of the points \( A, B \).
- III, 5. If for two triangles \( ABC \) and \( A'B'C' \) the congruences \( AB \equiv A'B', AC \equiv A'C', \angle BAC \equiv \angle B'A'C' \) hold, then the congruence \( \angle ABC \equiv \angle A'B'C' \) also holds.
- IV. (Euclid’s Axiom) Let \( a \) be any line and \( A \) a point not on it. Then there is at most one line in the plane, determined by \( a \) and \( A \), that passes through \( A \) and does not intersect \( a \).

The other axioms refer to 3-dimensional space or lay out details that we will just assume. We will assume the following, which basically say that lines behave like geodesics on a surface.

- Lines look like continuous curves that lie in (or on) a surface.
- Distances along a line make sense in the way we would expect.
- We can always follow a line any distance we choose in either direction. We will not assume that this implies that lines are infinitely long. In particular, a circle would satisfy this criterion.
- Angles can be measured, and they behave as we would expect.

We shall proceed as follows. We will show that if the geodesics on a surface satisfy Hilbert’s axioms I-1, I-2, III-5, and IV, then we must really be talking about a flat plane like the \( xy \)-plane. Doing this, we will be able to hit on the major ideas leading up to the discovery of non-Euclidean geometry.

1.1. **Two points determine a line.** Axioms I-1 and I-2 take Euclid’s first postulate and state explicitly that two distinct points contain exactly one line (no more, and no less). Note that this is not true in spherical geometry, and we will make specific use of these axioms, especially the second one.

1.2. **Side-Angle-Side.** Axiom III-5 should look a little like the SAS criterion of congruence. Euclid assumed this as obvious, and an axiom system is supposed to state everything explicitly. It’s also supposedly good form to state things in the most basic and minimal manner possible. So while Hilbert could have just used SAS here, he used a weaker (yet sufficient) statement instead.

Let’s prove SAS here. We have two triangles \( \triangle ABC \) and \( \triangle A'B'C' \), and to show that they are congruent, we want all three pairs of sides congruent and all three pairs of angles congruent. We won’t worry about area, so congruence of triangles means that the sides and angles are congruent.

Suppose \( AB \equiv A'B', AC \equiv A'C', \) and \( \angle BAC \equiv \angle B'A'C' \) (I’m just going to use the =-sign). Axiom III-5 says that \( \angle ABC \equiv \angle A'B'C' \). Now, the labeling of the vertices are arbitrary. We could just as easily have labeled them with \( B \) and \( C \) reversed. The axiom would then tell us that the other pair of angles are also
congruent. Therefore, we must also have that $\angle ACB = \angle A'C'B'$. We now have that all three pairs of corresponding angles are congruent, and two pairs of sides are congruent. All that is left is the pair $BC$ and $B'C'$.

**Theorem 1.** Given the conditions of Axiom III-5, it follows that all three pairs of corresponding angles are congruent.

Remember that we’re starting from scratch here, so we don’t have any theorems. We’re going to suppose that $BC < B'C'$ and try to get a contradiction. This would be sufficient, since if the inequality were pointing the other way, we could just put the primes on the first triangle. This means that there is a point $D'$ on (the interior) of segment $B'C'$ so that $B'D' = BC$. Applying Axiom III-5 (or the above theorem) to the triangles $\triangle ABC$ and $\triangle A'B'D'$, we must have that $\angle BAC = \angle B'A'D'$. We already had that $\angle BAC = \angle B'A'C'$, so the lines $A'C'$ and $A'D'$ must be the same lines (one of our general assumptions about angles). Now we have the lines $A'C'$ and $B'C'$ both containing the points $D'$ and $C'$. Axiom I-2 would force these two lines to be the same line, and we don’t really have a triangle. That’s our contradiction, and we can therefore conclude that $BC = B'C'$.

**The Side-Angle-Side Theorem.** In Hilbert’s axiom system, given triangles $\triangle ABC$ and $\triangle A'B'C'$, if $AB = A'B'$, $AC = A'C'$, and $\angle BAC = \angle B'A'C'$, then the two triangles are congruent. That is, all three pairs of corresponding sides and angles are congruent.

1.3. **Euclid’s Axiom.** Hilbert’s Axiom IV, which he calls Euclid’s Axiom, is a restatement of Euclid’s Fifth Postulate. It is sometimes called *Hilbert’s Axiom of Parallelism, Playfair’s Axiom,* or some variation of these.

Note that Axiom IV says that there is at most one parallel through $A$. This leaves open the possibility that a parallel does not exist. The main reason for this is that Hilbert is trying to use the weakest (most general) axioms that he can. In Euclidean geometry, we will want a parallel to exist, but this can be proven from the other axioms. Therefore, since Hilbert does not need an axiom for the existence of parallels, he does not include one. We do, however, need an axiom that makes sure that there aren’t too many parallels, and this is probably the most interesting and important issue arising from Euclid’s and Hilbert’s axiom systems.

Let’s show here that Hilbert’s Axiom IV follows from Euclid’s Fifth Postulate. Later on, we will show that the Fifth Postulate follows from Axiom IV (in the context of Hilbert’s other axioms), and so these two statements are actually equivalent.

So suppose we have a line $a$ and a point $A$ not on $a$. Pick any point $P$ on $a$, and the unique line $t$ through $P$ and $A$ will be called a transversal. Consider any other line $b$ through $A$. For convenience, pick points $Q$ on $a$ and $R$ on $b$ so that $Q$ and $R$ are on the same side of $t$. Now consider $x = \angle QPA + \angle PAR$. If $x \neq 180^\circ$, then Euclid’s Fifth Postulate tells us that $b$ intersects $a$, and so $a$ and $b$ would not be parallel. By the uniqueness of angles (a basic assumption), there is only one possible line $b$ that would make $x = 180^\circ$. Therefore, this one line is the only possible parallel (and even it doesn’t have to be parallel). This proves that Axiom IV follows from the Fifth Postulate, and Hilbert could have just as easily have used the Fifth Postulate as his Axiom IV.

2. **If $\alpha + \beta = 180^\circ$, then $l$ and $m$ are parallel**

OK. Hilbert used what he called Euclid’s Axiom to take the place of Euclid’s Fifth Postulate, but as we’ve seen, they say essentially the same thing. In particular, Euclid’s Axiom guarantees that we don’t have two parallels through the point $A$. Neither states that parallels actually exist, however. The reason for this is that we can prove the existence of parallels from the other axioms. Let’s do that now.

We want to prove that in the configuration of Figure 1, if $\alpha + \beta = 180^\circ$, then $l$ and $m$ are parallel. This means that $l$ and $m$ will never intersect.

We will assume the opposite. Suppose that $\alpha + \beta = 180^\circ$ and $l$ and $m$ do intersect. Therefore, there is a point of intersection $C$. Let’s assume that $C$ lies to the right. Consider a new point $C'$ that lies on $m$ to the left so that the distance from $B$ to $C'$ is the same as the distance from $A$ to $C$. 
We have that $AC = BC'$ by definition. Segments $AB = BA$, because they’re the same segment. Since
\[ \alpha + \beta = 180^\circ \]  
We also have that (only assuming that straight angles measure $180^\circ$)
\[ \beta + \beta' = 180^\circ \]
Subtracting equations gives us
\[ \alpha - \beta' = 0^\circ, \]
and
\[ \alpha = \beta'. \]
By SAS, the triangle $\triangle ABC$ is congruent to $\triangle BAC'$. Therefore, $\beta = \alpha'$. It follows that $\alpha + \alpha' = 180^\circ$. We are assuming that angles behave as we expect, so this means that the line $AC'$ must be the same as $l$. This forces us to conclude that $l$ and $m$ have two points in common. If $B \not\in l$, which is what we assume in Euclid’s Axiom, then $l$ and $m$ must be distinct lines, and we’ve got our contradiction. Our original assumption that $l$ and $m$ intersect is wrong, and therefore, $l$ and $m$ must, in fact, be parallel.

Note that the number $180^\circ$ comes from how we measure angles in degrees. No matter what units of measure we use for angles, the relevant thing is that the number that corresponds to two right angles (or one straight angle) goes here. We will see that the statement “the angle sum of a triangle is the same as the measure of a straight angle” is equivalent to the Fifth Postulate and Euclid’s Axiom.

2.1. Quiz.

-1– Did we use the Fifth Postulate or Euclid’s Axiom in this proof?
-2– Did we use the fact that the angle sum of a triangle is $180^\circ$?
-3– Did we use the fact that the measure of a straight angle is $180^\circ$?

3. Neutral geometry

You may have heard of Fermat’s Last Theorem (I say “fur MAH” or “fair MAH”). It was one of the big unsolved problems in mathematics until fairly recently, and these big problems drive a lot of the development in mathematics. Although mathematics was different then, we might say that the big problem between Euclid’s time and the time of Gauss, Bolyai, and Lobachevski (early 1800’s) was the Prove-Euclid’s-Fifth-Postulate Problem. Gauss, Bolyai, and Lobachevski finally realized that you can’t prove Euclid’s Fifth Postulate, and that’s when things started to get interesting. Translations of the papers written by Bolyai and Lobachevski are in the appendices of [Bonola].
I want to continue exploration of this Prove-Euclid’s-Fifth-Postulate Problem for awhile. For convenience, we will talk about these things within Hilbert’s axiom system, even though this came much later. Hilbert’s axiom system has an axiom called Euclid’s Axiom, as we’ve seen. **Euclid’s Axiom** (or Hilbert’s **Parallel Axiom**) is equivalent to Euclid’s **Fifth Postulate**, so if you’re talking about one, you’re also talking about the other. If you take Hilbert’s axiom system and throw out Euclid’s Axiom, we have an axiom system for something called **neutral geometry**. A traditional **modern geometry** course might spend two-thirds of a semester on neutral geometry. We won’t, because I don’t think it’s that important. It is a little bit important, however.

The Prove-Euclid’s-Fifth-Postulate Problem, therefore, can be stated this way:

> Prove Euclid’s Axiom as a theorem in neutral geometry.

### 3.1. Quiz.

- 1. State Euclid’s Axiom.
- 2. Is it actually possible to prove Euclid’s Axiom as a theorem in neutral geometry?
- 3. We will eventually find that the statement “the angle sum of a triangle is 180°” is equivalent to Euclid’s Axiom. Based on what I’ve said so far, is it possible to prove this statement as a theorem in neutral geometry?
- 4. What is neutral geometry?

If Hilbert’s axiom system is an axiom system for Euclidean geometry, and we could prove Euclid’s Axiom as a theorem in neutral geometry, then neutral geometry and Euclidean geometry would be the same. We know that Euclidean geometry satisfies all the axioms of neutral geometry, so if they’re not the same, then there must be at least one other geometry that also satisfies the axioms of neutral geometry. What Gauss, Bolyai, and Lobachevsky did was find another geometry. That geometry is usually called **hyperbolic geometry**, and today, we’re still talking about it as the other geometry. That’s not good. There are other geometries. On the other hand, hyperbolic geometry is an important geometry. We’ll talk about that later.

In the previous section, we showed the following is a theorem in neutral geometry.

> If in Figure 2 \(\angle 2 + \angle 4 = 180^\circ\), then \(l\) is parallel to \(m\).

Since all the axioms of neutral geometry are also axioms in Hilbert’s Euclidean geometry, we must be able to prove this statement in Euclidean geometry, as well. Therefore, this last statement is a theorem of Euclidean geometry.
The contrapositive of a statement is equivalent logically, so the following is also a theorem in neutral geometry (and therefore also in Euclidean geometry).

If in Figure 2 \( l \) is not parallel to \( m \), then \( \angle 2 + \angle 4 \neq 180^\circ \).

Neither of these statements is equivalent to \textbf{Euclid’s Fifth Postulate}, which can be phrased as follows.

If in Figure 2 \( \angle 2 + \angle 4 \neq 180^\circ \), then \( l \) is not parallel to \( m \).

We have not established this as a theorem in neutral geometry, and we will see that this is impossible. This statement, therefore, is only true in Euclidean geometry, and not in neutral geometry. Note that the contrapositive of this statement is also only true in Euclidean geometry.

If in Figure 2 \( l \) is parallel to \( m \), then \( \angle 2 + \angle 4 = 180^\circ \).

3.2. Quiz.

-1– In Figure 2, suppose \( \angle 2 = 150^\circ \) and \( \angle 4 = 27^\circ \). In Euclidean geometry, can you determine whether \( l \) is parallel to \( m \)? How about in neutral geometry?

-2– In Figure 2, suppose \( \angle 2 = 100^\circ \) and \( \angle 4 = 80^\circ \). In Euclidean geometry, can you determine whether \( l \) is parallel to \( m \)? How about in neutral geometry?

-3– In Figure 2, suppose \( \angle 1 = 100^\circ \) and \( l \) is parallel to \( m \). In Euclidean geometry, can you determine the measure of \( \angle 3 \)? How about in neutral geometry?

4. The \textbf{Alternate Interior Angles Theorem}

Given two lines \( l \) and \( m \) cut by a transversal \( t \), we showed earlier that if the interior angles on one side of the transversal sum exactly to 180°, then \( l \) and \( m \) are parallel. We did not use \textbf{Euclid’s Axiom} (or any of the other equivalent parallel axioms). Therefore, this would be a theorem of neutral geometry.

4.1. Quiz.

-1– Look at Figure 2. Suppose \( \angle 1 \cong \angle 4 \). You may assume that a straight angle measures 180°. What is \( \angle 1 + \angle 2 \)?

-2– What is \( \angle 2 + \angle 4 \)?

-3– In neutral geometry, what can you conclude about \( l \) and \( m \)? How do you know?

You have proved a version of the interior angles theorem that is true in neutral geometry.

\textbf{The Alternate Interior Angles (implies parallel) Theorem}. (neutral geometry) \textit{Given two lines \( l \) and \( m \) cut by a transversal \( t \) and alternate interior angles are equal, then \( l \) and \( m \) are parallel.}

In neutral geometry, whenever alternate interior angles are congruent (equal), we know that \( l \) and \( m \) are parallel. Since we are not assuming Euclid’s Axiom, it’s \textit{conceivable} that we could tilt \( m \) a little bit, and \( m \) would still be parallel to \( l \), but the alternate interior angles would no longer be congruent after tilting \( m \). In neutral geometry, therefore, we cannot conclude that parallel lines imply congruent alternate interior angles. This sort of dead end is a recurring outcome in the history of geometry leading up to the discovery of non-Euclidean geometry.
4.2. Quiz.

–1– In neutral geometry, if \(l\) and \(m\) are cut by a transversal, and alternate interior angles are congruent, must \(l\) and \(m\) be parallel?

–2– In neutral geometry, if \(l\) and \(m\) are cut by a transversal, and \(l\) and \(m\) are parallel, must alternate interior angles be congruent?

We generally equate congruent alternate interior angles with parallelism. Parallelism implies congruent alternate interior angles only in Euclidean geometry, however, and we need Euclid’s Fifth Postulate or one of its variations to prove this. It’s simplest to use Hilbert’s Euclid’s Axiom.

The logic goes like this. Suppose we have two lines \(l\) and \(m\) cut by a transversal \(t\), and \(l\) and \(m\) are parallel. Let’s let \(P\) be the point where \(m\) and \(t\) intersect. We want to show that alternate interior angles are congruent. Even if they aren’t congruent, we can construct another line \(m'\) through \(P\) such that alternate interior angles are congruent. In neutral geometry, and therefore also in Euclidean geometry, \(m'\) is parallel to \(l\). In Euclidean geometry, we also have Euclid’s Axiom, which says that there is at most one line through \(P\) that is parallel to \(l\). This means that \(m\) and \(m'\) must be the same line, and so the alternate interior angles associated with \(m\) must be congruent.

The (parallel implies) Alternate Interior Angles Theorem. (Euclidean geometry) Given two lines \(l\) and \(m\) cut by a transversal \(t\) and that \(l\) and \(m\) are parallel, then alternate interior angles are congruent.

![Figure 3](image-url)

**Figure 3.** The lines \(l\) and \(m\) are cut by a transversal \(t\).

4.3. Quiz.

–1– In Figure ??, the pair \(\angle 5\) and \(\angle 8\) are called alternate exterior angles. Prove that if \(\angle 5 = \angle 8\), then \(l\) and \(m\) are parallel.

–2– The pair \(\angle 6\) and \(\angle 7\) are also called alternate exterior angles. Is the fact that alternate interior angles are congruent imply that \(l\) and \(m\) are parallel a theorem in neutral geometry or only in Euclidean geometry?

–3– If \(l\) and \(m\) are parallel, must alternate exterior angles be congruent in neutral geometry?

–4– In Euclidean geometry?

–5– The pair \(\angle 1\) and \(\angle 7\) are called corresponding angles. Prove that if \(\angle 1 = \angle 7\), then \(l\) and \(m\) are parallel.

–6– The pair \(\angle 2\) and \(\angle 8\) are also corresponding angles. The pair \(\angle 3\) and \(\angle 5\) and the pair \(\angle 4\) and \(\angle 6\) are also. If corresponding angles are congruent, can we conclude that \(l\) and \(m\) are parallel in neutral geometry?

–7– How about in Euclidean geometry?

–8– If \(l\) and \(m\) are parallel, can we conclude that corresponding angles are congruent in neutral geometry?
We are accustomed to assuming that the three angles of a triangle always sum to 180° (or π radians). We will see, however, that the angle sum of a triangle is closely related to the various parallel axioms, and once we give these away, the number 180° goes away also.

Using our neutral geometry Alternate Interior Angles Theorem, we can try to prove that the angle sum of a triangle is 180°.

**Figure 4.** The lines \( l \) and \( m \) are parallel to the base of the triangle \( AB \).

In Figure 4 is a triangle \( \triangle ABC \). We can construct lines \( l \) and \( m \), as in the picture, so that \( \angle 1 = \angle 1' \) and \( \angle 2 = \angle 2' \). According to the Alternate Interior Angles Theorem, we can conclude that \( l \) is parallel to line \( AB \), and we can also conclude that \( m \) is parallel to line \( AB \). In **Euclidean geometry**, we have Euclid’s *Axiom* (or the *Fifth Postulate*), which says that through \( C \) there can be at most one line parallel to \( AB \). In Euclidean geometry, therefore, \( l \) and \( m \) must be the same line, and therefore, \( \angle 1', \angle 3, \) and \( \angle 2' \) together form a straight angle, which measures 180°. It would follow that \( \angle 1 + \angle 2 + \angle 3 = 180° \).

In **neutral geometry**, however, there may be multiple parallel lines through \( C \), and so \( l \) and \( m \) may be different lines. If this is the case, then \( \angle 1' + \angle 3 + \angle 2' \neq 180° \).

6. **What Euclid Did**

If you read Euclid’s *Elements* carefully (I haven’t), you might get the feeling that Euclid suspected the existence of something like hyperbolic geometry. For example, people have noticed that he seems to try to avoid use of the Fifth Postulate. Therefore, he has many neutral geometry results among his propositions. Let’s follow Euclid’s investigation of angle sums.

When two lines cross, they form four angles, and the opposite pairs are called **vertical angles**. Consider **Proposition 15** on page 277 of Euclid’s *Elements*. Here Euclid shows that vertical angles are congruent (e.g., \( \angle AED \cong \angle CEB \)). This result follows from the assumption that straight angles measure 180°.

Euclid’s **Proposition 16** states that, given a triangle, an exterior angle is always bigger than each of the interior angles opposite of it. I have drawn Figure 5 to look similar to the one in the *The Elements*, but the letters are switched around.

In Figure 5, we’re starting with the triangle \( \triangle ABC \). The side \( AB \) is extended to a point \( D \). The angle \( \angle CBD \) is an **exterior angle**, and the angles \( \angle BAC \) and \( \angle ACB \) are the angles opposite. Note that

\[
\angle ABC + \angle CBD = 180°
\]
Let $E$ be the midpoint of side $BC$. I’ve put single slash marks through the segments $CE$ and $BE$ to indicate they’re congruent. We can then draw a segment through $A$ and $E$ to a point $F$ so that $E$ is the midpoint of $AF$. Then we connect the points $B$ and $F$ with a line segment. Note that $\angle 4$ and $\angle 4'$ are vertical angles, so they must be congruent.

From the SAS criterion (which is one of Hilbert’s axioms other than Euclid’s axiom, so is in neutral geometry), triangles $\triangle AEC$ and $\triangle FEB$ are congruent. In particular, $\angle 3 = \angle 3'$. Since $\angle CBD = \angle 3' + \angle 6$, we clearly have that

$$\angle CBD > \angle ACB,$$

and the exterior angle is larger than one of the opposite angles. I’ll let you show that this exterior angle is also larger than the other opposite angle.

6.1. Quiz.

-1– Draw a new picture without $E$ and $F$. You still should have the exterior angle $\angle CBD$. Now extend side $CB$ downwards to a point $G$. This will form $\angle ABG$ as an exterior angle.
-2– Your new picture is kind of like a mirror image of Figure 5. Applying what we just figured out, which opposite angle is smaller than this exterior angle, $\angle ABG$?
-3– OK. How are angles $\angle ABG$ and $\angle CBD$ related?
-4– How are we doing with Proposition 16?

7. The Sum of Two Angles

Finally, let’s nail down Euclid’s Proposition 17. This one says that any two angles of a triangle sum to less than 180°.

Let’s go back to the triangle $\triangle ABC$ with the extension to $D$. We know that

$$\angle ACB < \angle CBD.$$  \hspace{1cm} (7)

We also know that

$$\angle ABC + \angle CBD = 180^\circ.$$  \hspace{1cm} (8)

It follows that

$$\angle ABC + \angle ACB < 180^\circ.$$  \hspace{1cm} (9)

There isn’t anything special about which two angles we’re talking about, so clearly, any pair of angles from this triangle must sum to less than 180°.
As a look ahead, note that in Figure 5, \( AC \) and \( BF \) are parallel and congruent. Therefore, \( \triangle ABC \) and \( \triangle ABF \) have the same base and the same height. The areas are the same, but the angle at \( A \) is approximately cut in half. Imagine repeating this construction on \( \triangle ABF \). The angle at \( A \) gets smaller still, but the area stays the same. Keep going. What’s special about the triangles as you go out in this sequence?

**References**


