1. A Model for Spherical/Elliptic Geometry

Let’s take an overview of what we know.

Hilbert’s axiom system goes with Euclidean geometry. This is the same geometry as the geometry of the \(xy\)-plane. Mathematicians have come to view Hilbert’s axioms as a bunch of very obvious axioms plus one special one, Euclid’s Axiom. The specialness of Euclid’s Axiom is partly historical, but there are good mathematical reasons, as well.

**Euclid’s Axiom.** Given a line \(l\) and a point \(P\) not on \(l\), there is at most one line through \(P\) that is parallel to \(l\). *(We could have said exactly one line parallel here)*

Euclid’s Axiom in the context of neutral geometry is equivalent to the statement, “The angle sum of a triangle is 180°.” We’re going to embrace this alternative as we move on. We can draw a picture of a triangle, but we can’t draw parallel lines (showing the entire length of the lines). With parallel lines, the important stuff happens (or doesn’t happen) outside of the picture, and we’ve kind of started to see that what happens outside of the picture is not as obvious as we think.

One big transition in the history of geometry occurred when Gauss, Lobachevski, and Bolyai realized that we could pull Euclid’s Axiom and replace it with something like the hyperbolic axiom, and we would still have a consistent axiom system.

**The Hyperbolic Axiom.** Given a line \(l\) and a point \(P\) not on \(l\), there are at least two lines through \(P\) that are parallel to \(l\). *(There are actually two asymptotic parallels and infinitely many more parallels in between.)*

This axiom is equivalent to the statement, “The angle sum of a triangle is less than 180°.” In fact, and we’re going to run with this, the angle sum of a triangle (in radians) plus the area of the triangle always equals \(\pi\).

In the context of these two parallel axioms, we have exactly one parallel in Euclidean geometry and infinitely many parallels in hyperbolic geometry. There is a third possibility that we have not yet considered. No parallels at all.

**The Elliptic Axiom.** Given a line \(l\) and a point \(P\) not on \(l\), there are no lines through \(P\) that are parallel to \(l\). *(This can be stated more directly. There are no parallel lines.)*

Now, we can’t just attach the elliptic axiom to neutral geometry, since we showed that there are parallels in neutral geometry. It is possible, however, to make small changes to the axioms of neutral geometry so that the resulting axiom system for elliptic geometry is consistent. We won’t go there, however. One of the big problems with Hilbert’s axiom system is that we can’t easily move from one geometry to another.

Here, we’ll look at two elliptic geometries. We’ll look at spherical geometry mostly, and in passing, the elliptic geometry that goes with the elliptic axiom system just mentioned. Each of these spaces is sometimes called the geometry of Riemann, but according to Bonola (page 147 in *Non-Euclidean Geometry*), it is not clear which geometry Riemann actually had in mind. From Riemann’s point of view, in fact, they are simply different spaces with the same geometry, so he may have actually been talking about both.

2. The Models for Elliptic Geometry

Axioms will begin fade into the background for us now, and what a synthetic geometer (i.e., an axiomatic geometer) might regard as only a model, we’ll embrace as a geometric space. One of these spaces is the sphere, which consists of the points that lie on the surface of the unit sphere,

\[
S = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}.
\]

Very important: When I say sphere, I will always mean the surface of the sphere. The interior points may lie inside the sphere, but they are not part of the sphere. I will use the word **ball** for the sphere plus the interior
points. The lines in spherical geometry are the great circles. Great circles are those circles on the sphere that have the same radius as the sphere (in this case \( r = 1 \)). They also divide the sphere exactly in half. The equator is a great circle, and any circle passing through both the north pole and south pole is a great circle. There are many more lines, of course. We can immediately see how this cannot be a neutral geometry. Lines are not infinite in length, although they can be continued indefinitely\(^1\). Also, every pair of great circles intersects in two points\(^2\). The other elliptic geometry eliminates this conflict with the Hilbert-like axioms. The projective plane is the same as the sphere, except antipodal points are identified. Antipodal points are pairs of points on the sphere that are exactly opposite each other (e.g., the north and south pole are antipodal points). By identify, I mean that we think of the two antipodal points as being a single point. With this identification, great circles now only intersect in a single point just as lines intersect in Euclidean and hyperbolic geometry. The projective plane is a really interesting space to visualize, but we won’t worry about that now. Just remember that the projective plane is a weird space with the same local geometry as the sphere, and it is a model for the synthetic elliptic geometry.

Note that the sphere is the actual object we’re interested in. In its geometry, distances are measured along the surface directly, and angles are measured as they appear. We can visualize spherical geometry in Euclidean space, because elliptic geometry is in some sense smaller than Euclidean geometry. In fact, if we try to squeeze the Euclidean plane onto the sphere, we’d have to make it really wrinkly. Squeezing the hyperbolic plane down onto the Euclidean plane without deforming distances would make it really wrinkly, as well.

3. **Lines in spherical geometry**

To get a feeling for spherical geometry, get a ball and put a rubber band around the middle. It should look like the equator. If you have a world globe, it actually has the equator marked on it. This is a great circle. The horizontal circles (the latitude lines) are not great circles except for the equator. Note that they are all different sizes. The longitude lines (which pass through the north and south poles) are great circles. Given any great circle, you can roll the ball around so that it sits in the equator’s position.

Any plane that passes through the center point of the sphere will intersect the sphere in a great circle.

3.1. **Quiz.**

- \(^1\) Given any three distinct points in \( xyz \)-space, how many planes contain all three points? There are two cases.
- \(^2\) Given two points on the sphere, how many planes pass through these two points and the center of the sphere? There are two cases.
- \(^3\) Given two points on the sphere, how many great circles contain both points? There are two cases.

4. **Area on the sphere**

Recall that the area of a sphere is given by \( A = 4\pi r^2 \), so the area of the unit sphere is \( 4\pi \). You should remember this number. It is very important.

Look at Figure 1. We have two great circles passing through the north and south poles and intersecting at right angles. These great circles divide the sphere into four equal slices called lunes. Since the total area is \( 4\pi \), the area of each of these lunes must be \( \pi \). Note that each lune is bounded by segments of two great circles, which are the lines in this geometry. Therefore, we could say that a lune is bounded by a 2-gon (in the sense that a triangle is a 3-gon). Furthermore, the angles on each of these 2-gons is \( \frac{\pi}{2} \) (a right angle).

\(^1\) Compare this to Euclid’s **Second Postulate**

\(^2\) Compare this to Euclid’s **First Postulate**
Figure 1. Not a great picture, but try to see two great circles passing through the north and south poles and intersecting at right angles.

Now consider three great circles through the north and south poles, which form six equal angles at each pole. We have six lunes, each with area $\frac{4\pi}{6} = \frac{2\pi}{3}$. Each angle from the bounding 2-gons measures $\frac{2\pi}{6}$.

With four great circles and eight equal lunes, we get areas of $\frac{\pi}{2}$ and angles $\frac{\pi}{4}$.

In general, suppose we have a lune with angles equal to $\theta$ (for convenience, imagine the vertices at the north and south poles). There are $2\pi$ radians around the north pole, and $\theta$ occupies some portion of these. The area of the lune makes the same proportion to the total area of the sphere. In other words,

\begin{equation}
\frac{\theta}{2\pi} = \frac{\text{area}}{4\pi},
\end{equation}

and the area of a lune is

\begin{equation}
\text{area} = 2\theta.
\end{equation}

Figure 2. Three great circles on the sphere forming a triangle. There is an identical triangle on the other side.

5. The Angle Sum of a Triangle

As you might suspect, the angle sum of a triangle is not 180°, nor is it $\pi$ radians. You may also suspect that the angle sum of a triangle in spherical geometry is related to the triangle’s area. Look at the triangle shown in Figure 2. It has angles measuring $\alpha$, $\beta$, and $\gamma$. Note that there is an identical triangle on the other side. We can find an equation relating the area and angle sum as follows.
The angle sum of this triangle is $\alpha + \beta + \gamma$. Since the sides appear to be bowed out, we should expect this angle sum to be bigger than $\pi$ radians or $180^\circ$ (the opposite of hyperbolic geometry).

The great circles in pairs form pairs of lunes. One pair has angles equal to $\alpha$, one pair has angles $\beta$, and the third pair has angles $\gamma$. These six lunes completely cover the sphere, although there is some overlap. In particular, each pair of lunes cover both copies of the triangle, so each triangle is covered three times. The rest of the sphere is covered only once. Let’s let $A$ be the area of each triangle. We now have that the total area of the sphere is equal to the sum of the areas of the lunes, minus the four extra coverings of the triangles. In other words,

$$4\pi = 2 \cdot 2\alpha + 2 \cdot 2\beta + 2 \cdot 2\gamma - 4A.$$  

This simplifies to

$$\pi = \alpha + \beta + \gamma - A$$

and

$$\alpha + \beta + \gamma = \pi + A.$$  

**Theorem 1.** In spherical geometry, (measured in radians) the angle sum of a triangle minus its area is always equal to $\pi$.

This result should shock you. We’ve known since elementary school that the **angle sum of a triangle** had something to do with $180^\circ$, and when we learned about radians, also something to do with $\pi$. Now, we see that in hyperbolic geometry and in elliptic geometry, the difference between the angle sum of a triangle and $\pi$ is equal to the area of the triangle. In terms of **synthetic geometry**, we would think that the three geometries were fundamentally different, and if anything, Euclidean geometry and hyperbolic geometry were more closely related.

This relationship is a special case of something called the **Gauss-Bonnet theorem**. In its various forms, the Gauss-Bonnet theorem captures the essence of 2-dimensional geometry in a way that the various forms of Euclid’s Fifth Postulate could never do.

### 5.1. Quiz.

1. Draw a picture, as best you can, of two great circles through the north and south poles that meet at right angles. Add the equator. You should get eight triangles. What is the angle sum and area of each triangle. Does this agree with Theorem 1?

2. We have to measure the angles in radians to use theorem 1. What does it say about angle sums in degrees? In other words,

$$\alpha + \beta + \gamma - \text{what?} = 180^\circ.$$  

**References**

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