Rules for Fair Divisions

We are going to look at sharing “games,” where players will try to fairly divide a “cake” among themselves. Here are the basic assumptions behind what the different methods must satisfy.

- There is no “mom,” who will step in and figure out a way that’s best for everyone.
- A fair share for a given player means that that player views the value of his/her share to be at least one-nth of the total value when there are n players.
- Each player doesn’t know (or care) what the values of the other players’ shares are.
- The total value of the cake is the same for each player.

Example. Let’s suppose we have a $24 cake that is half strawberry and half chocolate. Let’s also suppose that we have three players, Adam, Betty, and Carl. Adam likes strawberry and chocolate equally. Betty likes chocolate, but absolutely hates strawberry. Carl likes chocolate twice as much as strawberry.

When we cut a cake, we’ll always use radial cuts, and the pieces will be wedges. We’ll generally refer to them as $s_1$, $s_2$, $s_3$, etc., for slice 1, slice 2, slice 3, etc. Let’s look at the values of the pieces from the point of view of each player.

Adam. To Adam, the value of strawberry and the value of chocolate is the same, and so, the strawberry half and the chocolate half each have half of the total value, or $12. The values of our three slices, therefore, must be

1. $s_1 = \$12$
2. $s_2 = \$6$
3. $s_3 = \$6$

A fair share to Adam would be worth $24/3 = \$8$, so he would only view $s_1$ as being a fair share.
Betty. To Betty, who hates strawberry, sees the strawberry half as having no value, and so the chocolate half alone must be worth the entire $24. The values of our three slices are

(5) Betty
(6) \[ s_1 = 0 \]
(7) \[ s_2 = 12 \]
(8) \[ s_3 = 12 \]

Again, since there are three players, a fair share must be worth at least $8 to Betty, and so she would see both \( s_2 \) and \( s_3 \) as being fair shares.

Carl. Since Carl likes chocolate twice as much as strawberry, the strawberry half must have one-third of the total value, or $8, and the chocolate half is worth $16 to him. The values of the three slices to Carl, therefore, are

(9) Carl
(10) \[ s_1 = 8 \]
(11) \[ s_2 = 8 \]
(12) \[ s_3 = 8 \]

Here, all three slices would be fair shares, as far as Carl is concerned.

The fair divisions. Looking at the values, there are basically two ways that the slices can be distributed so that each of the players gets a fair share.

(1) Adam gets \( s_1 \), Betty gets \( s_2 \), and Carl gets \( s_3 \).
(2) Adam gets \( s_1 \), Betty gets \( s_3 \), and Carl gets \( s_2 \).

These will both be considered fair divisions, since each player sees their piece as having at least one-third of the total value (because there are three players). That’s it. That will be our only consideration.

Note that we will assume that Adam and Carl don’t care that Betty thinks she got a $12 slice. Note also that the individual values add up to more than $24. These are both things that we will expect to happen in our problems, and we won’t worry about them.

Example. Same cake, players, and preferences as the previous example, but now, we’re going to cut the cake differently. See the picture below. If I draw a picture that looks symmetric, and I don’t indicate otherwise, you can assume that it is symmetric. Therefore, those two little chocolate wedges must measure 30° (since they need to add up with the 120° wedge to make 180°).
Adam. To Adam, the strawberry half and the chocolate half each are worth $12. Those 30° wedges must have value

\[ \frac{30}{180} \cdot 12 = 2 \text{ dollars}, \]

and the 120° wedge must have value

\[ \frac{120}{180} \cdot 12 = 8 \text{ dollars}. \]

The values of our three slices, therefore, must be

\[ s_1 = 6 + 2 = 8 \text{ dollars,} \]
\[ s_2 = 6 + 2 = 8 \text{ dollars,} \]
\[ s_3 = 8 \text{ dollars,} \]

We could have seen these values by remembering that chocolate and strawberry have the same value for Adam, and these three pieces all measure 120°. In any case, all three slices are fair shares for Adam.

Betty. For Betty, the strawberry half has no value, and the chocolate half is worth $24. Those 30° wedges are worth

\[ \frac{30}{180} \cdot 24 = 4 \text{ dollars,} \]

and the 120° wedge is worth

\[ \frac{120}{180} \cdot 24 = 16 \text{ dollars.} \]

The values of our three slices are

\[ s_1 = 0 + 4 = 4 \text{ dollars,} \]
\[ s_2 = 0 + 4 = 4 \text{ dollars,} \]
\[ s_3 = 16 \text{ dollars,} \]

Only \( s_3 \) is a fair share.

Carl. To Carl, the strawberry half is worth $8, and the chocolate half is worth $16. The 30° chocolate wedges are worth

\[ \frac{30}{180} \cdot 16 = 2.67 \text{ dollars,} \]

and the 120° wedge is worth

\[ \frac{120}{180} \cdot 16 = 10.67 \text{ dollars.} \]

The values of the three slices to Carl, therefore, are

\[ s_1 = 4 + 2.67 = 6.67 \text{ dollars,} \]
\[ s_2 = 4 + 2.67 = 6.67 \text{ dollars,} \]
\[ s_3 = 10.67 \text{ dollars.} \]

Just as with Betty, only \( s_3 \) is a fair share to Carl.

The fair divisions. Looking at the values, there is no fair division with these cuts, since both Betty and Carl need to have \( s_3 \).
Practice Problems

1. Same cake, players, and preferences as the examples in this lecture. Find the values of each of the slices to all three players, and determine all possible fair divisions.

2. We’re looking at a $12 cake that is half strawberry and half chocolate. Our three players are Alice, Bob, and Carol. Alice likes strawberry twice as much as chocolate. Bob likes chocolate twice as much as strawberry. Carol likes strawberry three times as much as chocolate.

Find the values of each of the slices to each of the players for the cuts shown below. Determine all fair divisions.

1) Adam: $9, $6, and $9. Betty: $6, $0, and $18. Carl: $8, $4, $12. There are no fair divisions, since $s_2$ is not fair to anyone.

2) Alice: $4.67, $4.67, and $2.67. Bob: $3.33, $3.33, and $5.33. Carol: $5, $5, and $2. Fair divisions: Alice gets $s_1$, Bob gets $s_3$, and Carol gets $s_2$ is one fair division. The other is Alice gets $s_2$, Bob gets $s_3$, and Carol gets $s_1$. 