

Monday, April 23, 2012.

Objectives: Find the explicit formula for the Fibonacci sequence, and look at some instances of the Fibonacci sequence.

DERIVING BINET'S FORMULA

OK. So I showed you the explicit formula for the Fibonacci sequence several lectures ago. It is usually called *Binet's formula*, although Binet probably wasn't the first to figure it out. The first may have been Euler, who we've talked about before, but it's hard to know for sure, who was the very first to know something. Both Binet and Euler are famous mathematicians, and part of why I want to show you this stuff is that a lot of things that famous mathematicians do aren't always that hard to understand. It's not trivial, but you can do it.

Here we go. The Fibonacci sequence is the sequence

$$(1) \quad 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

that satisfies the relational equation

$$(2) \quad A_n = A_{n-1} + A_{n-2}.$$

We know how to find explicit formulas for relational equations like this. We guess at an explicit formula of the form

$$(3) \quad A_n = Cx^n,$$

and plug it into the relational equation. We get

$$(4) \quad Cx^n = Cx^{n-1} + Cx^{n-2}.$$

This always simplifies to an equation that has the same coefficients as the relational equation. In this case, we get the quadratic equation

$$(5) \quad x^2 = x + 1.$$

To use the quadratic formula, we move the terms to the left side, and get

$$(6) \quad x^2 - x - 1 = 0.$$

This makes $a = 1$, $b = -1$, and $c = -1$, and so the quadratic formula gives us the two solutions

$$(7) \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Since 5 doesn't have a nice square root, we'll just leave it in this form. What this tells us is that there are two possible explicit formulas taking the form

$$(8) \quad A_n = C \left(\frac{1 + \sqrt{5}}{2} \right)^n \quad \text{and} \quad C \left(\frac{1 - \sqrt{5}}{2} \right)^n,$$

and we also have seen that some combination of these might be the explicit formula we want. In particular, all of the explicit formulas will take the form

$$(9) \quad A_n = B \left(\frac{1 + \sqrt{5}}{2} \right)^n + C \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

We want the values for B and C that correspond to the first two terms $A_0 = 1$ and $A_1 = 1$.

THE MESSY PART

You don't have to know this part in detail, but take a look at the computations. They aren't *that* bad.

So we know both A_0 and A_1 two different ways, in terms of the explicit formula and also their actual values.

$$(10) \quad A_0 = B \left(\frac{1 + \sqrt{5}}{2} \right)^{(0)} + C \left(\frac{1 - \sqrt{5}}{2} \right)^{(0)} = 1$$

$$(11) \quad A_1 = B \left(\frac{1 + \sqrt{5}}{2} \right)^{(1)} + C \left(\frac{1 - \sqrt{5}}{2} \right)^{(1)} = 1.$$

Simplifying these gives us two equations.

$$(12) \quad B + C = 1$$

$$(13) \quad B \left(\frac{1 + \sqrt{5}}{2} \right) + C \left(\frac{1 - \sqrt{5}}{2} \right) = 1.$$

Now, multiply the second equation by 2 to get

$$(14) \quad B + C = 1$$

$$(15) \quad B(1 + \sqrt{5}) + C(1 - \sqrt{5}) = 2.$$

If you multiply the B and C through, in the second equation, and collect the $\sqrt{5}$'s, you get

$$(16) \quad B + C = 1$$

$$(17) \quad (B + C) + (B - C)\sqrt{5} = 2.$$

We know that $B + C = 1$, so that second equation is really

$$(18) \quad 1 + (B - C)\sqrt{5} = 2,$$

and so

$$(19) \quad (B - C)\sqrt{5} = 1,$$

and therefore,

$$(20) \quad B - C = \frac{1}{\sqrt{5}}.$$

Our pair of equations, therefore, is

$$(21) \quad B + C = 1$$

$$(22) \quad B - C = \frac{1}{\sqrt{5}}.$$

Adding these two equations together give us

$$(23) \quad 2B = 1 + \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} + \frac{1}{\sqrt{5}},$$

and dividing both side by 2 tells us what B is,

$$(24) \quad B = \frac{\sqrt{5} + 1}{2\sqrt{5}}.$$

We can now go back to any of our equations to find C . One of these equations is $B + C = 1$, and so

$$(25) \quad C = 1 - B = 1 - \frac{\sqrt{5} + 1}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} - \frac{\sqrt{5} + 1}{2\sqrt{5}} = \frac{\sqrt{5} - 1}{2\sqrt{5}}.$$

Whew!

Writing our $\sqrt{5}$'s second, our explicit formula becomes

$$(26) \quad A_n = \frac{1 + \sqrt{5}}{2\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1 - \sqrt{5}}{2\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n,$$

Which simplifies to

$$(27) \quad A_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1}.$$

This is Binet's formula.

CHECKING ON BINET

OK. Let's check this. It's a little easier to work with decimal approximations than the square roots, so Binet's formula is approximately equal to

$$(28) \quad A_n = \frac{(1.618)^{n+1} - (-0.618)^{n+1}}{2.236}.$$

Recall that the Fibonacci sequence starts off

$$(29) \quad 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

and $A_7 = 21$. Binet's formula states that

$$(30) \quad A_7 = \frac{(1.618)^8 - (-0.618)^8}{2.236} = \frac{46.971 - .021}{2.236} = 20.997$$

This is not quite 21, but it's close. The difference is due to round-off error.

1. PRACTICE PROBLEMS

Use Binet's formula to find

1. A_5 .
2. A_{10} .

Answers are on the last page.

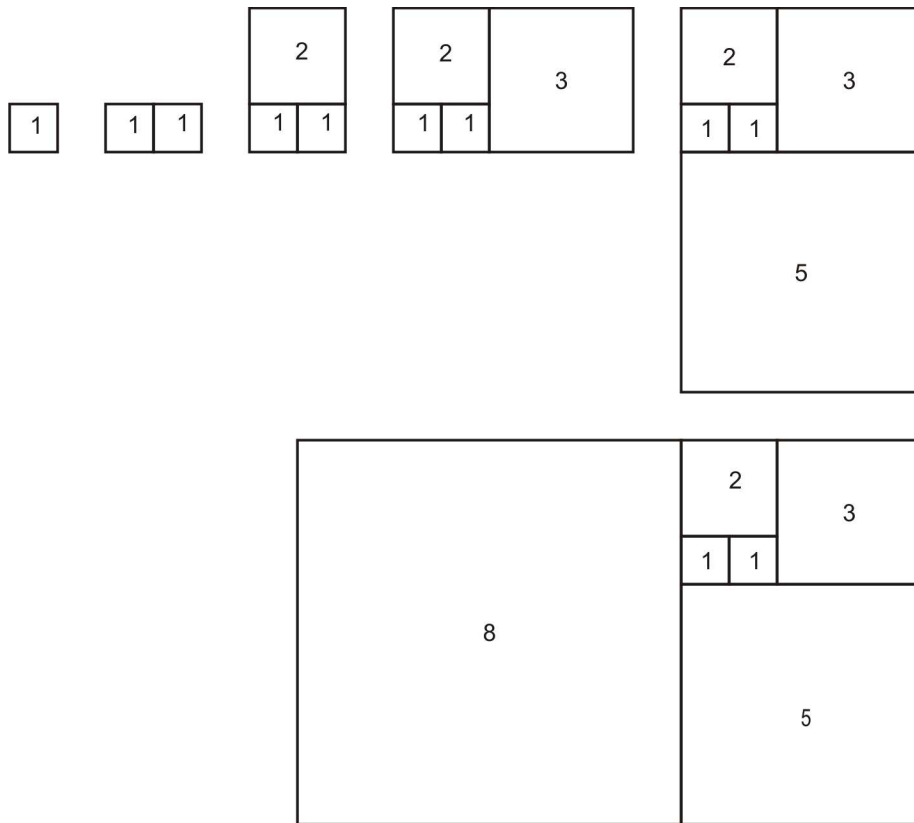
THE GOLDEN RATIO

Note that in Binet's formula, $0.618 < 1$, so powers of this get smaller and smaller. The powers of 1.618 are the main parts of the formula. This number

$$(31) \quad \frac{1 + \sqrt{5}}{2} \approx 1.618$$

is called the *golden ratio*. The ancient Greeks liked this number. For example, they liked to use *golden rectangles*, rectangles whose ratio between long side and short side equaled the golden ratio, in their architecture. Properties of the golden ratio and golden rectangles give a little insight into why the Fibonacci sequence appears frequently in nature. For example, you can find the Fibonacci numbers by counting the scales on a pineapple, the rows of seeds in a sunflower, etc.

Let's start with this geometric example. We're going to start with a 1×1 square, add another 1×1 square to it to make a larger rectangle. After that, we will add a square to the long side of the rectangle to build a bigger rectangle. Look at the following pattern.



The long side of each rectangle is a Fibonacci number, and this continues indefinitely. Note that the ratio of long side to short side in the last rectangle is

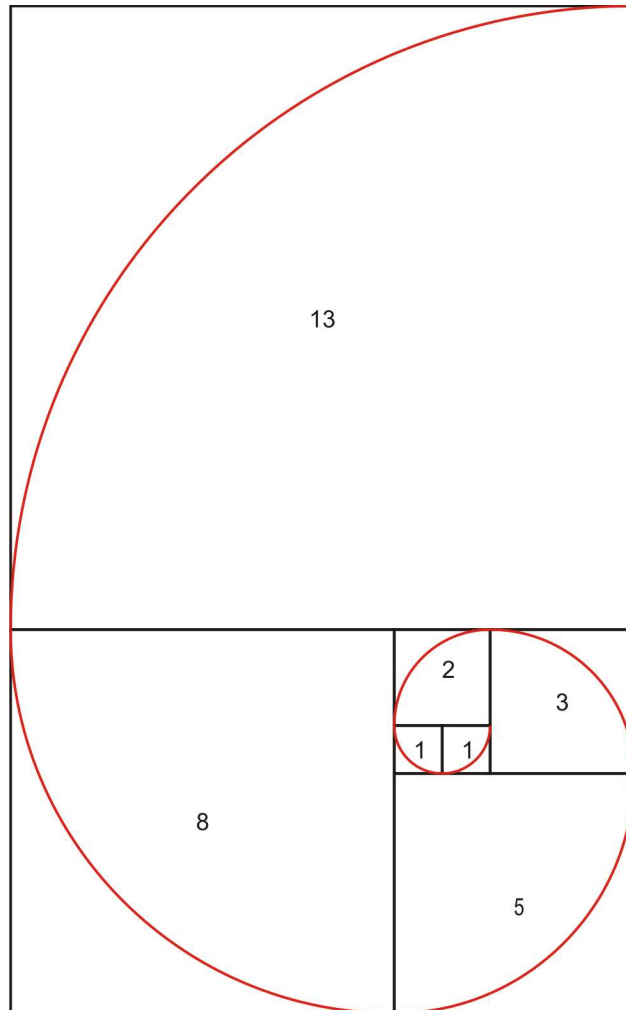
$$(32) \quad \frac{13}{8} \approx 1.625$$

If we kept making the rectangles, at some point we would have a 144 by 89 rectangle (two consecutive Fibonacci numbers), and their ratio is

$$(33) \quad \frac{144}{89} = 1.6179775 \dots \approx 1.618,$$

which is very close to the golden ratio. These rectangles get closer and closer to being perfect golden rectangles.

This suggests one property that golden rectangles have. If you add a square to the long side of a golden rectangle, you get another golden rectangle. This is a clue to why Fibonacci numbers appear in nature. This growth pattern is a way of getting bigger, maintaining the same basic overall shape, but without changing the old structure. We don't grow that way. As children grow, all their parts get bigger. Some animals and plants, however, grow this other way. Look at the next page.



Here I've inscribed a spiral into our rectangular growth pattern. Can you think of any animals that have this kind of shape? One such animal is the nautilus. As it grows, it needs a bigger shell, but it's hard to make a shell bigger by stretching it out. What it does instead is to simply add a chamber to its existing shell that's big enough for the mushy part of the nautilus to occupy. The old part of the shell is still there, and the overall shape is the same. It's just bigger now. This is called *gnomonic growth*, where you grow by adding a piece to your old shape, but still maintain the same basic shape. Adding squares to the long side of golden rectangles is a geometric example of gnomonic growth.

QUIZ 30 WORKSHEET

1. Our Fibonacci sequence starts off $A_0 = 1$, $A_1 = 1$, $A_2 = 2$, $A_3 = 3$, $A_4 = 5$, etc. What is A_{10} ?
2. What is A_{15} ?
3. Using the approximate formula (Equation 28), find A_2 . Round your answer correctly to three decimal places.
4. Use the approximate formula to find A_7 .
5. The explicit formula for the Fibonacci sequence has a name. What is that name?
6. The number $\frac{1+\sqrt{5}}{2}$ has a special name. What is that name?
7. If the length of the long side of a rectangle divided by the length of the short side gives you the number in Problem 6, what is the rectangle called?
8. If something grows by adding a new piece to the old structure while maintaining the same overall shape, we have a name for that kind of growth. What is it?
9. What animal did I mention as growing this way?
10. Give the approximate decimal representation of $\sqrt{5}$ rounded correctly to three decimal places.

Practice Problem Answers: 1) $A_5 = \frac{1.618^6 - (-0.618)^6}{2.236} = \frac{17.942 - .056}{2.236} = 7.999$, which is 8 with round-off error.

2) $A_{10} = \frac{1.618^{11} - (-0.618)^{11}}{2.236} = \frac{198.959 + 0.005}{2.236} = 88.982$, which is 89 with round-off error.