Lab 3

Question 1. In the plane, a triangle’s angles add up to $180^\circ$. Find a triangle on the cone from Lab 1 (formed by removing the third quadrant) that contains the vertex. What is the angle sum of this triangle? Find another triangle that contains the vertex, and find its angle sum. Find a triangle that does not contain the vertex and its angle sum. Any conjectures?

Question 2. We saw in Lab 2 that we can remove wedges measuring anywhere between 0 and $2\pi$ radians. We will say that at the cone-point formed by removing a $\theta$-wedge, the *impulse curvature* at the cone-point is $\theta$ (with $\theta$ being measured in radians, but we won’t give any units to the impulse curvature). Build a cone whose vertex has impulse curvature $\frac{\pi}{3}$ radians, and find a triangle containing the vertex. What is the angle sum of the triangle?

Instead of removing a wedge to form a cone-point, we can also *add* a wedge. When we remove a wedge, we’ll call the resulting cone-point an *elliptic cone-point*, and when we add a wedge, we’ll call the result a *hyperbolic cone-point*. For convenience, we’ll consider the impulse curvature at a hyperbolic cone-point to be negative. Specifically, adding a $\theta$-wedge introduces an impulse curvature of $-\theta$.

![Figure 1: Adding a wedge creates a hyperbolic cone-point.](image)

Question 3. In Figure 1, the $45^\circ$-wedge shown is to be inserted into the cut at $V$. The geodesics $a$ and $b$ continue into the wedge and intersect. The geodesics $a$, $b$, and $c$, therefore, meet to form a triangle. Find the angle sum of the triangle and the impulse curvature at $V$.

Question 4. Make a conjecture about the angle sum of a triangle that contains a cone-point with impulse curvature $\theta$. Do the same for a quadrilateral.

Question 5. Build a surface that has two cone-points, one with impulse curvature $I_1 = \frac{\pi}{2}$ and the other with $I_2 = -\frac{\pi}{4}$. Also, find a triangle that contains both cone-points. What is the angle sum for this triangle? Make a conjecture about the angle sum of a triangle containing $n$ cone-points with impulse curvatures $I_1$, $I_2$, $\ldots$, $I_n$. 
Lab 3 – Answers and Notes

Figure 2: An easy triangle on the 90°-cone.

**Question 1.** A really easy triangle to consider is shown in Figure 2. The two lines at the cut come together to form one side of the triangle, so the angles are \( \alpha \), \( \beta \), and \( \gamma \). These are all right angles, so the angle sum of this triangle is 270°. You should find that all triangles that contain the vertex of the cone will have an angle sum of 270°. The triangles that do not contain the vertex are regular plane triangles when you flatten the cone out, so these should have angle sums of 180°.

Figure 3: A triangle on a 60°-cone.

**Question 2.** To have an impulse curvature of \( \frac{\pi}{3} \), we need to remove a wedge of \( \frac{\pi}{3} \) radians or 60°. One triangle is shown in Figure 3. The angles \( \phi \) and \( \psi \) must add up to 180° so that the side that crosses the cut is a geodesic. As drawn, both of these angles measure 90°. For this particular triangle, \( \alpha = \beta = 75° \) and \( \gamma = 90° \), so the angle sum is 240°. Any triangle containing the vertex will also have an angle sum of 240°.

**Question 3.** The angles \( \alpha \) and \( \beta \) in Figure 1 must measure 45° in order for \( a \) and \( b \) to be geodesics. The quadrilateral bounded by the cuts and the geodesics \( a \) and \( b \) must therefore have angles measuring 45°, 135°, 135°, and \( \gamma \). This is a plane figure, so the angles must add up to 360°, and therefore, \( \gamma = 45° \). The angle sum of the triangle on this hyperbolic cone is 135° or \( \frac{3\pi}{4} \) radians. The impulse curvature at \( V \) is \( \frac{\pi}{4} \).

**Question 4.** All of the triangles we have seen so far have an angle sum of \( \pi \) radians plus the impulse curvature. The simplest generalization to a quadrilateral would be \( 2\pi \) radians plus the impulse curvature. Check a few cases to see if this works. It’s not terribly hard to prove that both of these are true.
Question 5. You should find that the angle sum of the triangle is \( \pi + \frac{\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4} \). In general, it is true that

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\text{angle sum} = \pi + \sum I_j.
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