Paper Models for Surfaces with Curvature

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Objectives

Geometric spaces can be manipulated.

Introduce geodesics.

Introduce total curvature.

Motivate Gauss-Bonnet Theorem.

Motivate 2-manifolds.
Notes

Labs consist of a series problems.

Three students worked together outside of class.

Students needed little help from me.

Accessible to undergraduates.
Sample Problem from Lab 2

Problem: Accurately describe a surface and two geodesics with exactly three points of intersection.

Note: Gauss-Bonnet theorem requires that each of the enclosed regions must have total curvature greater than $\theta$. 
Solution allowing cone points

Remove two wedges measuring $\psi$ radians.

Must have $\psi > \theta$.

Introduces two cone points with total curvature of $\psi$ at each.
The corresponding paper model

Continuation of geodesics are rotated by $\psi$ radians.
Gauss-Bonnet Theorem

Geodesic curvature $C$: $\kappa_g = \frac{1}{r}$.

Total geodesic curvature on $C$:

$$\int_C \kappa_g \, ds = \frac{1}{r} (2\pi r - \theta r) = 2\pi - \theta.$$ 

Compare to Gauss-Bonnet:

$$\int_C \kappa_g \, ds = 2\pi - \int_R K \, dA.$$
Sphere is tangent to the cone at $C$.

The geodesic curvatures for $C$ are the same.

Total curvature on the sphere is also $\theta$.

$$\int_D K \, dA = \int_D \frac{1}{R^2} \, dA = \cdots = \theta$$