MA 2232 Lecture 26 - Ratio Test

Wednesday, April 13, 2016.

Objectives: Introduce the ratio test.

Recall that a geometric series,

\[ \sum ar^i = a + ar + ar^2 + ar^3 + \cdots, \]

will converge only if \(-1 < r < 1\). We might write this as \(|r| < 1\). The thing to notice about the terms of a geometric series is that each term is \(r\) times the previous one. In other words,

\[ \frac{ar^{i+1}}{ar^i} = r. \]

Of course, a series doesn’t have to be entirely geometric to converge. If it becomes geometric after awhile, then we know how it will converge. Perhaps, even, if a series becomes more and more geometric, and becomes geometric in the limit, that should be enough. That turns out to be true.

**Ratio Test**

Given any series \(\sum a_i\), consider the ratio

\[ \frac{a_{i+1}}{a_i}. \]

**Case 1.** If this ratio is a constant, then we have a geometric series.

**Case 2.** If

\[ \lim_{i \to \infty} \left| \frac{a_{i+1}}{a_i} \right| = r, \]

then this series should converge very much like a geometric series with ratio \(r\).

**Case 2a.** If \(r = 0\), then our series should converge faster than a geometric series. In particular, it must converge.

**Case 2b.** If \(r < 1\), then our series should converge like a geometric series with ratio \(r < 1\). In particular it must converge.

**Case 2c.** If \(r = 1\), the our series is on the boundary between convergent and divergent series. In particular, the series may or may not converge.

**Case 2d.** If \(r > 1\), then the terms in our series are actually getting bigger, so the series must diverge.

**Basic Principle 1. Ratio Test** Given a series \(\sum a_i\) and

\[ \lim_{i \to \infty} \left| \frac{a_{i+1}}{a_i} \right| = r. \]

Then the series will

- **converge** if \(r < 1\),
- **diverge** if \(r > 1\).

The Ratio Test doesn’t tell us anything, if \(r = 1\).
Example. Consider the series
\[ \sum \frac{1}{i!} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots \]
We could compare this to the geometric series
\[ \sum \frac{1}{2^i} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots \]
While at first the powers of 2 are bigger than the factorials, the factorials quickly take off, because we’re multiplying larger numbers. The factorial series, therefore, must be much smaller. We can see this in the ratio test.

For the series
\[ \sum \frac{1}{i!} \]
we check the ratio
\[ \lim_{i \to \infty} \left| \frac{i!}{(i+1)!} \right| = \lim_{i \to \infty} \left| \frac{i!}{(i+1)i!} \right| = \lim_{i \to \infty} \left| \frac{1}{i+1} \right| = 0. \]
Since \( r = 0 < 1 \), the ratio test says that this series must converge, and we would expect this series to converge faster than any geometric series.

Example. Use the ratio test on
\[ \sum \frac{i!}{3^i} \]
Just looking at this, you can probably see that the terms go to infinity, so there’s no way this is going to converge, but let’s just do the ratio test anyway.

\[ \left| \frac{(i+1)!}{3^{i+1}} \right| = \frac{(i + 1)!}{3^{i+1}} \cdot \frac{3^i}{i!} = \frac{i + 1}{3} \to \infty = r. \]
We’ll say that \( r = \infty \) is bigger than 1, so our series diverges by the ratio test.

Example. Use the ratio test on
\[ \sum \frac{i}{2^i} \]
Here, we have
\[ \left| \frac{i+1}{2^{i+1}} \right| = \frac{i+1}{2^{i+1}} \cdot \frac{2^i}{i} = \frac{i+1}{2 \cdot i} \to \frac{1}{2} = r. \]
This \( r \) is less than 1, so our series converges by the ratio test.

**Quiz 26**

1. Use the ratio test on
\[ \sum \frac{(i!)^2}{(3i)!} \]

2. Use the ratio test on
\[ \sum \frac{(i!)^2}{(3i)!} \]
1. **Homework 26**

Use the ratio test on the following series. That is, find \( r \), and state the conclusion according to the ratio test (which might be “Don’t know.”)

1. \( \sum \frac{1}{i} \). (This is called the harmonic series.)
2. \( \sum \frac{1}{i^7} \).
3. \( \sum \frac{1}{i^7} \).
4. \( \sum \frac{2^i}{i^7} \).
5. \( \sum \frac{(2i)!}{(3i)!} \).

**Answers:**

1) \( \frac{i}{i+1} \rightarrow 1 \), so don’t know. (The harmonic series turns out not to converge.)

2) \( \frac{2(i+2)(2i+2)}{(3i+3)(3i+2)(3i+1)} \rightarrow 0 = r \).

3) \( \frac{1}{i} \rightarrow 0 \), so converge.

4) \( \frac{2}{i+1} \rightarrow 0 \), so converge.

5) \( \frac{(2i+2)(2i+1)}{(3i+3)(3i+2)(3i+1)} \rightarrow 0 = r \), so converge.