Objectives: Warm up to thinking about logical statements.

**True and False Statements**

In some sense, mathematics is a study of statements and whether these statements are true or false, or perhaps, neither. Just studying statements like this without significant further context is called *logic*, but all areas of mathematics are basically just logic applied to some context. You’ll sometimes hear people talk about an “advance” in thinking called *fuzzy logic*, where statements can be partly true, but fuzzy logic is really just regular logic applied to statements that can be partly true.

Truth in this context is relative. We’ll always be starting with a few statements that are simply *assumed* to be true, and then figuring out which other statements must also be true if those assumed statements are really true. Our processes will also have a few assumed truths to get us started.

This isn’t a super hard-core course in logic, so we’re not going to start with a minimum collection of axioms (the assumed true statements), but with a bunch of statements that we all would say are obviously true. For example, the statement

> The number 2 is an even number.

is an obviously true statement. An obviously false statement would be something like

> The number 3 is greater than the number 5.

Some statements are neither true or false. For example,

> The number $x$ is an even number.

might be true or might be false, depending on what $x$ is. Now, $x$ could be a specific number that we just don’t know, in which case, the statement really is true or false, but we don’t know which. On the other hand, $x$ might represent all the integers, in which case, the statement is sometimes true and sometimes false, depending on which integer $x$ is.

**If . . ., then . . . Statements**

Not knowing or not specifying a value for a variable does not preclude a statement from being true or false. For example, we might have the statement

> If $x$ is an integer multiple of 4, then $x$ is an even number.

This, we would probably consider as being obviously true. Just to simplify things, we’ll assume that variables will take the largest range of values that make sense (not necessarily truth) in that context. In this case, we would assume that $x$ is an integer, and that $x$ wouldn’t be something like a tree.

The various values of $x$ in this case fall into one of three categories:

*If true, then true.*: For example, $x = 8$ makes the If-part true, since

- 8 is a multiple of 4

is a true statement. The Then-part is also true:

- 8 is an even number.
This is the main part of an If-Then statement. We want all the values of $x$ that make the If-part true to also make the Then-part true.

**If false, then true.** : For example, $x = 6$ makes the If-part false, since

6 is a multiple of 4

is a false statement. It’s perfectly fine that the Then-part is true

6 is an even number.

**If false, then false.** : The last case occurs for numbers like $x = 3$. Here,

3 is a multiple of 4

is false, as is

3 is an even number.

This is fine too. An If-Then statement makes no assertions on the cases where the If-part is false.

**Basic Principle 1.** An If-Then statement, like the one above, is True, if all values of $x$ fall into these three categories: If true, then true. If false, then true. If false, then false.

The one “bad” case that will make an If-Then statement false is the “If true, then false” case. For example, consider the statement

If $x$ is a multiple of 3, then $x$ is a multiple of 2.

This is a false statement. Somewhere in your mind, you’re probably thinking that there are multiples of 3 that aren’t multiples of 2. For example, if $x$ is 9, we have the true statement

9 is a multiple of 3

and the false statement

9 is a multiple of 2.

We’ll call $x = 9$ a **counter example**. The existence of a counter example is what makes an If-Then statement false. In fact, let’s make that the definition.

**Definition 1.** Let $P(x)$ and $Q(x)$ be statements involving a variable $x$, like the ones we’ve been playing with. The statement

If $P(x)$, then $Q(x)$

is False, if there is a value for $x$, say $x = a$, such that $P(a)$ is True and $Q(a)$ is False. Such a value $x = a$ will said to be a **counter example**. The statement “If $P(x)$, then $Q(x)$” is True, if there are no counter examples.

**Satisfied Vacuously**

Our definition brings up an uncomfortable consequence. Consider the statement

If $x$ is an odd multiple of 2, then $x$ is a multiple of 5.
Since there are no odd multiples of 2, this is a somewhat strange statement, because the If-part is always false. Since the If-part is always false, there can be no counter examples. No counter examples means that our definition makes this If-then statement True. A statement like this is said to be satisfied vacuously. Whenever the If-part is vacuous, the If-then statement is automatically True.

Do you like that? Probably not. Mathematicians have chosen to make vacuously-true statements True, as a matter of convenience. Things work out better. For example, it makes our definition simple. We also will like to have the $x$'s that make the If-part True to be a subset of the $x$'s that make the Then-part True. It’s also convenient to allow the empty set to be a subset of any set. This definition is consistent with that.

**Homework 01**

1. Consider the following set of statements, and determine if they are True or False. You don’t have to give a reason.

   For all of the following, assume that $x$ ranges over the entire set of integers (i.e., \( \ldots, -2, -1, 0, 1, 2, 3, \ldots \)).

   a. If $x$ is an even integer, then $x$ is an integer.

   b. If $x$ is an integer, then $x$ is an even integer.

   c. If $x$ is not an integer, then $x$ is an even integer.

   d. If $x$ is a multiple of 3, then $x$ is a multiple of 6.

   e. If $x$ is a multiple of 6, then $x$ is a multiple of 3.

   f. If $x$ is not a multiple of 3, then $x$ is not a multiple of 6.

   g. If $x$ is not a multiple of 6, then $x$ is not a multiple of 3.

   h. If $x$ is a negative integer times $-3$, then $x$ is a negative integer.

   i. If $x$ is not a negative integer, then $x$ is not a negative integer times $-3$.

   j. If $x$ is a square root of an integer, then $x$ is not 3.

2. The proof required to show that an If-Then statement is false, is simple. You just find a counter example (and show that it actually is a counter example, if not obvious). Proving an If-Then statement is true is generally harder. For the statements in Problem 1 that are False, give a counter example. You don’t have to show that it actually is a counter example.

3. Which of the statements in Problem 1 are satisfied vacuously?

Answers on next page.
1: True: a, c, e, and f. False: b, d, g, h, i, and j.

2: There are lots of counter examples, I’m only giving one for each. b: \( x = 1 \). d: \( x = 3 \). g: \( x = 3 \). h: \( x = 3 = (-1)(-3) \). i: \( x = 3 \). j: \( x = 3 \).

3: Only c.