For convenience, we’ll sometimes want to use shorthand symbols to represent statements. For example, we might have

\[ P(x) = \text{The integer } x \text{ is a multiple of 4}. \]

We’re actually talking about a bunch of statements. One of these is \( P(8) \), which is a True statement, and another is \( P(5) \), which is False. I want to say that \( P(x) \) is sometimes True and sometimes False. Given a statement containing a variable, which is actually a collection of statements, we can create a single statement in several different ways. There are things called quantifiers that consider all the statements together, and also our If-Then statement, which contains a quantifier.

**For every.** One quantifier is *for every* (also *for all* or *for each*). Used on the statement \( P(x) \) above, we get the statement

For every integer \( x \), \( x \) is a multiple of 4.

Generally, it’s obvious that we’re only talking about integers, or only talking about real numbers, so we’ll just say “for every \( x \).” Using the shorthand \( P(x) \), we could say

For every \( x \), \( P(x) \).

The symbol \( \forall \) is often used as a symbol for “for every.” We can then say

\[ \forall x, P(x). \]

Not surprisingly, the statement \( \forall x, P(x) \) is True, if \( P(x) \) is True for every single \( x \). Otherwise, it’s False. Since there are \( x \)'s that are not multiples of 4, this particular statement is False.

**There exists.** The other commonly used quantifier is *there exists* (also *for some* or *there is*). On our statement \( P(x) \) above, we have

There exists an integer \( x \), such that \( x \) is a multiple of 4.

The standard symbol for “there exists” is \( \exists \), so we could also write

\[ \exists x, P(x). \]

The statement \( \exists x, P(x) \) is True, if there is at least one \( x \) that makes \( P(x) \) True. Since \( P(8) \) is True, that’s enough to make \( \exists x, P(x) \) a True statement.

**Quiz 02A**

Suppose \( Q(x) = \text{The integer } x \text{ is a real number} \). Determine if the following are True or False.

1. \( \forall x, Q(x) \).
2. \( \exists x, Q(x) \).

**And’s, Or’s, and Not’s**

Before we get back to the If-Then statement, let me introduce three basic logical operators.
And. Given two statements $P$ and $Q$, the statement “$P$ and $Q$” is True, if both $P$ and $Q$ are True. Otherwise, it’s False. For example, using the statement $P(x) = \text{“}x\text{ is a multiple of } 4\text{”}$, as above, we can talk about the statement

(4) $P(8)$ and $P(12)$.

Both are true, so this is a True statement. On the other hand, the statement

(5) $P(8)$ and $P(9)$

is False.

Or. Given two statements $P$ and $Q$, the statement “$P$ or $Q$” is True, if at least one of $P$ and $Q$ is True. It is False otherwise. For example, using the same $P(x)$ as above,

(6) $P(8)$ or $P(12)$

is True. So is

(7) $P(8)$ or $P(9)$

True. On the other hand,

(8) $P(3)$ or $P(5)$

is False.

Not. Finally, we’ll have the operator “Not $P$,“ which is True when $P$ is False, and False when $P$ is True. I’ll use the shorthand symbol $\sim P$ to represent Not $P$. With the same $P(x) = x$ is a multiple of 4 as above,

(9) $\sim P(8)$

would be False, and

(10) $\sim P(3)$

would be True.

Back to If-Then

Our If-Then statements have a built in quantifier. Letting

(11) $P(x) = \text{The integer } x \text{ is a multiple of } 4$,

and

(12) $Q(x) = \text{The integer } x \text{ is a multiple of } 2$,

we have the If-Then statement

If $P(x)$, then $Q(x)$.

The symbol $\Rightarrow$ is often used to indicate If-Then, so we can write this as

(13) $P(x) \Rightarrow Q(x)$,

and we might also say, “$P(x)$ implies $Q(x)$.” These all say the same thing as

If $x$ is a multiple of 4, then $x$ is a multiple of 2.

We would look at this, and conclude that this statement is clearly True. But note that even though there is a variable in this statement, it is a single statement that is True or False (in this case True), and this is based on knowing what happens with all the $x$’s.
**Equivalent versions to** $P(x)\Rightarrow Q(x)$. In general, an If-Then statement of the form $P(x)\Rightarrow Q(x)$ is True, if all of the $x$’s fall into the three “good” categories: $P(x)$ is True and $Q(x)$ is True, $P(x)$ is False and $Q(x)$ is True, and $P(x)$ is False and $Q(x)$ is False. In other words,

\begin{equation}
P(x)\Rightarrow Q(x) \equiv (P(x) \text{ and } Q(x)) \text{ or } (\neg P(x) \text{ and } Q(x)) \text{ or } (P(x) \text{ and } \neg Q(x)).
\end{equation}

We can simplify this a bit by noting that if we have $\neg P(x)$, the $Q(x)$ doesn’t matter. So now we have

\begin{equation}
P(x)\Rightarrow Q(x) \equiv (P(x) \text{ and } Q(x)) \text{ or } \neg P(x).
\end{equation}

This simplifies further when we examine the possibilities. If $\neg P(x)$ is True, then we’re good. If $\neg P(x)$ is False, then $P(x)$ is True, but we still need $Q(x)$ to be True. So really, all we need is either $\neg P(x)$ or $Q(x)$ to be True. In other words,

\begin{equation}
P(x)\Rightarrow Q(x) \equiv \neg P(x) \text{ or } Q(x).
\end{equation}

Following a different line of reasoning, saying that all the $x$’s fall into the three “good” categories, is equivalent to saying that none of the $x$’s fall into the one “bad” category: $P(x)$ is True and $Q(x)$ is False. This gives us another equivalent form for an If-Then,

\begin{equation}
P(x)\Rightarrow Q(x) \equiv (P(x) \text{ and } \neg Q(x)).
\end{equation}

If you look at the last two equivalent forms, you may notice that the $\neg$ kind of looks like it distributes. In particular, we have

\begin{equation}
\neg(P(x) \text{ and } \neg Q(x)) \equiv \neg P(x) \text{ or } Q(x),
\end{equation}

and the $\neg$ distributes over the and, except that the and changes to an or. If you think about it, that should make sense. This property is one of what we call the De Morgan’s Laws.

**De Morgan’s Laws.** There are two basic De Morgan’s Laws. One is

\begin{equation}
\neg(P \text{ and } Q) \equiv \neg P \text{ or } \neg Q,
\end{equation}

and the other is

\begin{equation}
\neg(P \text{ or } Q) \equiv \neg P \text{ and } \neg Q.
\end{equation}

**Homework 02**

1. State whether the following are True or False. You don’t have to give a reason. Let $P(x) = x$ is an even number, and let $Q(x) = x$ is a multiple of 6.

   a. $\neg P(3)$.
   
   b. $\neg P(12)$ or $Q(12)$.
   
   c. $P(x)\Rightarrow Q(x)$.
   
   d. $Q(x)\Rightarrow P(x)$.
   
   e. $P(x)\Rightarrow P(x)$.
   
   f. $P(7)$ and $\neg Q(11)$.

2. Now, $P(x)$ and $Q(x)$ are just general statements about $x$. Determine whether the following are True or False (or there’s not enough information).

   a. $(P(x) \text{ and } \neg P(x))\Rightarrow Q(x)$.
   
   b. $(P(x) \text{ or } \neg P(x))\Rightarrow (Q(x)$ and $\neg Q(x))$.
   
   c. $P(x)\Rightarrow (P(x) \text{ or } Q(x))$.  

3. Do the following using the equivalent forms of an If-Then statement or DeMorgan’s Laws.

a. Write “\(P(x) \Rightarrow \sim Q(x)\)” as an or statement.

b. Write “\(P(x) \Rightarrow \sim Q(x)\)” as a negated and statement.

c. Write “\(P(x) \text{ and } Q(x)\)” in terms of an \(\Rightarrow\) statement.

Answers on next page.
1a: T
b: T
c: F
d: T
e: T
f: F

2a: T (satisfied vacuously)
b: F (the If-part is always True, and the Then-part is always False).
c: T (any $x$ that makes $P(x)$ True will also make $P(x)$ or $Q(x)$ True).

3a: $\sim P(x)$ or $\sim Q(x)$.
b: $\sim(P(x)$ and $Q(x))$.
c: $\sim(P(x)\Rightarrow\sim Q(x))$. 