MA 3260 Lecture 04 - Stuff about numbers

Thursday, September 4, 2014.

Objectives: Investigate basic properties of numbers.

Let me start by naming our basic sets of numbers. In the context of sets, \( x \in A \) will mean that \( x \) is an element of the set \( A \).

**Natural/Counting Numbers.** \( \mathbb{N} = \{1, 2, 3, \ldots\} \).

**Integers.** \( \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \).

**Rational Numbers.** \( \mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\} \).

**Real Numbers.** \( \mathbb{R} \) is the set of points on the real number line.

**Irrational Numbers.** \( \mathbb{I} \) is the set of real numbers that are not rational.

**Complex Numbers.** \( \mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\} \).

**Basic Assumed Facts.**

We will assume that we understand the basic ordering properties of \( \mathbb{Z} \) and \( \mathbb{R} \). We won’t try to prove basic statements about \( \neq, <, =, >, \leq, \text{ and } \geq \). For example, if \( z \in \mathbb{Z} \), we’ll consider it obvious that there are six integers strictly between \( z \) and \( z + 7 \), and that \( z < z + 7 \).

We will assume basic arithmetic and algebra. For example, if \( x \in \mathbb{R} \), we’ll assume that \( \frac{14}{7} = 2x \) without justification.

**A NUMBER FACT**

If \( x < y \) are two rational numbers, then there is another rational number \( z \), such that \( x < z < y \).

Let \( x < y \in \mathbb{Q} \). This means that \( x = \frac{a}{b} \) and \( y = \frac{c}{d} \) for integers \( a, b, c, \text{ and } d \) with \( b \neq 0 \) and \( d \neq 0 \). Consider the number \( \frac{x+y}{2} \). Since \( x < y \), we know that

\[
(1) \quad x = \frac{x + x}{2} < \frac{x + y}{2} < \frac{y + y}{2} = y,
\]

so \( \frac{x+y}{2} \) lies between \( x \) and \( y \). Is it rational? Note that

\[
(2) \quad \frac{x + y}{2} = \frac{\frac{a}{b} + \frac{c}{d}}{2} = \frac{ad + bc}{2bd}.
\]

Since \( a, b, c, d \in \mathbb{Z}, ad + bc \in \mathbb{Z}, \text{ and since } b \neq 0 \text{ and } d \neq 0, \text{ then } 2bd \neq 0 \). Therefore, \( \frac{x+y}{2} \) is rational.

**QUIZ 04A**

1. Show that if \( x < y \in \mathbb{R} \), then there is a \( z \in \mathbb{R} \), such that \( x < z < y \). You may use the fact that sums, products, quotients, etc. of reals are real (as long as you do not divide by zero).

2. Explain with an example why a similar statement cannot be made about integers.

3. Is there a smallest natural number?

4. Is there a smallest positive rational?
INTEGER STUFF

A lot of the structure of the integers revolves around the concept of dividing evenly.

Definition 1. The phrase “m is a multiple of n” means that there is an integer q ∈ Z such that m = qn. We will also use the phrase “n divides m” to mean the same thing, and use the symbols m|n. We’ll also say that n is a factor of m.

Of course, there’s only so much dividing evenly we can do.

Definition 2. An integer p greater than 1, is prime, if the only positive integers that divide p are p and 1.

We don’t know what all the primes are exactly, but clearly we can list them out in order

\[ p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, \ldots \]

We’ll take it as a fact that every positive integer z has a unique prime factorization

\[ z = 2^{k_1}3^{k_2}5^{k_3}7^{k_4}11^{k_5} \ldots p_n^{k_n}. \]

Greatest Common Divisors, GCD’s

A divisor is basically the same as a factor. Given two integers, a common divisor is an integer that divides both. The biggest common divisor is called the greatest common divisor, also known as a GCD.

For example, consider 12 and 18. The common divisors are 1, 2, 3, and 6. The biggest common divisor is the GCD, 6. We’ll write

\[ \text{GCD}(12, 18) = 6. \]

A linear combination of 12 and 18 is an integer that can be expressed in the form

\[ s \cdot 12 + t \cdot 18, \]

where s and t are integers. So,

\[ (-4) \cdot 12 + (5) \cdot 18 = 42 \]

is a linear combination of 12 and 18.

Quiz 04

Find the smallest positive linear combination of 12 and 18.

Observation. Note that \( s \cdot 12 + t \cdot 18 = 6(s \cdot 2 + t \cdot 3) \), so 6 divides any linear combination of 12 and 18. This means that the smallest positive linear combination can’t be any smaller than 6. As long as you can get 6, that’s the best you can do.

Homework 04

If you read the observation after the Quiz, it seems that the smallest linear combination can’t be any smaller than the GCD. The question is can you always get down to the GCD?

1. Find the GCD, and a linear combination of the two numbers equal to the GCD.
   a. 3 and 5.
   b. 10 and 15.
   c. 12 and 15.
2. Weird Fact: The primes are basically independent with respect to multiplication. For example, take every positive integer’s prime factorization, change the ordering on the primes, and all your multiplication facts will stay the same. Let me explain with an example.

Our standard ordering on the primes is 2, 3, 5, 7, 11, . . .

Let’s change the order of the first three, and leave the rest the same, 3, 5, 2, 7, 11, . . .

We take a number like $12 = 2^2 3^1 5^0 7^0 \cdots$. If I change the ordering of the primes, I would get $3^2 5^1 2^0 7^0 \cdots = 45$. In other words, in the prime factorization, I change 2’s to 3’s, 3’s to 5’s, and 5’s to 2’s. I’ll write $T(12) = 45$.

Another example, $5 = 2^0 3^0 5^1 \rightarrow 3^0 5^0 2^1 = 2$, so $T(5) = 2$.

So we have $T(12) = 45$ and $T(5) = 2$. I said that multiplication facts stay the same. By that I mean

$$T(12 \cdot 5) = T(12) \cdot T(5).$$

In particular,

$$60 = 12 \cdot 5 \rightarrow 45 \cdot 2 = 90,$$

and so $T(60)$ should be 90. Is it?

$$60 = 2^2 3^1 5^1 \rightarrow 3^2 5^1 2^1 = 90.$$

a. Transform the numbers 5, 14, and 70. What is $T(5) \cdot T(14)$?

b. Transform the numbers 6, 15, and 90. What is $T(6) \cdot T(15)$?

Answers: 1a) $(-3)3 + (2)5 = 1$. b) $(-1)10 + (1)15 = 5$. c) $(-1)12 + (1)15 = 3$. d) $(9)9 + (-4)20 = 1$.

2a) $T(5) = 2$, $T(14) = 21$, $T(70) = 42$.

b) $T(6) = 15$, $T(15) = 10$, $T(90) = 150$. 