We have a machine that is set to run for $x$ hours, turn itself off for 3 hours, and then restart. Starting at 10 o’clock this process is repeated 7 times, and at the end of the last 3-hour rest period, it is 5 o’clock. What integers can $x$ be? In other words,

(1) \[ 10 \text{ o’clock} + 7(x + 3) = 5 \text{ o’clock}. \]

Find all integer solutions for $x$.

There are more interesting applications of what I’m about to propose, but at least we’ll have something to talk about.

**Integers Modulo 12**

Let’s solve this equation “on the clock.” By this, I mean that we only work with the hours on the clock, the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and instead of 12, I’ll go back to 0. So we’re saying now that $13 = 1$, 14 = 2, etc. I sometimes say that mathematics is the systematic exploitation of ambiguity, and that’s what we’re doing here. For example, $\frac{1}{2}$ and $\frac{2}{4}$ are different things, but thinking of them as being the same makes fractions more useful.

**Integers modulo 12.** Mathematically, clock arithmetic is doing algebra in $\mathbb{Z}_{12}$, the integers modulo 12. The numbers in $\mathbb{Z}_{12}$ are the integers, except they’re all replaced by the remainder we get after dividing by 12.

For example, $39 = 3 \cdot 12 + 3$, so in $\mathbb{Z}_{12}$, $39 = 3$. Since the remainders are less than 12, $\mathbb{Z}_{12}$ has 12 elements

(2) $\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

We’ll have addition and multiplication that kind of looks like regular addition and multiplication, but only with these 12 elements. For example,

(3) $6 + 9 = 15 = 3$,  
so $6 + 9 = 3$ in $\mathbb{Z}_{12}$. Multiplication works the same way

(4) $3 \cdot 7 = 21 = 9$.

Before going further, let’s solve our equation in $\mathbb{Z}_{12}$,

(5) $10 + 7(x + 3) = 5$.

Subtract 10 from both sides to get

(6) $7(x + 3) = 7$.

We’ll have to be careful about the next step, divide both sides by 7, but in this case it’s OK,

(7) $x + 3 = \frac{7}{7} = 1$.

Finally, subtract 3 from both sides.

(8) $x = 10$.

Going back to the regular integers, $x = 10, 22, 34, 46, 58, \ldots$ If we don’t mind negative time periods, we also have $-2, -14, \ldots$. 

1
Subtraction

We take subtraction for granted. If we want to understand it better, it’s useful to try to look at subtraction in an unfamiliar setting. You would study such things in modern algebra (a.k.a. abstract algebra). One way of interpreting subtraction is as follows. If we want to know what $7 - 5$ is, we might give the answer a name, like $x$, and consider the equation

\[(11) \quad 7 - 5 = x.\]

We could then define the subtraction as

\[(12) \quad 7 = x + 5,\]

and ask what number plus 5 is equal to 7? In this case, $x = 2$.

An alternative interpretation would make subtracting 5 equivalent to adding a negative 5. So

\[(13) \quad 7 - 5 = 7 + (-5).\]

In $\mathbb{Z}_{12}$, $-5 = 7$, so adding $-5$ is the same as adding 7, so

\[(14) \quad 7 - 5 = 7 + (-5) = 7 + 7 = 14.\]

There are two basic properties at work here. We need that $5 + (-5) = 0$, and $x + 0 = x$. The property that adding zero does essentially nothing is generally said by mathematicians as “zero is an additive identity.” If two elements add together to equal an additive identity, we say that they’re additive inverses.
But of course. With subtraction in $\mathbb{Z}_{12}$, we can also just do the subtraction as regular integers, and then convert to an element of $\mathbb{Z}_{12}$, like
\[(15) \quad 4 - 9 = -5 = 7.\]

**Division**

Division is to multiplication as subtraction is to addition. In multiplication, 1 is a multiplicative identity, since we have
\[(16) \quad x \cdot 1 = x\]
in both $\mathbb{Z}_{12}$ and multiplication with regular numbers. In the real numbers, $\mathbb{R}$, we have multiplicative inverses like $\frac{1}{3} \cdot 3 = 1$. In $\mathbb{Z}_{12}$ we only have
\[(17) \quad 1 \cdot 1 = 1, \quad 5 \cdot 5 = 1, \quad 7 \cdot 7 = 1, \quad \text{and} \quad 11 \cdot 11 = 1.\]
In other words, in $\mathbb{Z}_{12}$, 1, 5, 7, and 11 are their own multiplicative inverses. Dividing by these elements of $\mathbb{Z}_{12}$ is straightforward. For example,
\[(18) \quad 8 \div 7 = 8 \cdot 7 = 8.\]
We can also interpret the division as “What times 7 equals 8?” In other words, we want to figure out
\[(19) \quad 8 \div 7 = x,\]
and think of this as
\[(20) \quad 8 = x \cdot 7.\]
Looking at the multiplication table, we find 8 in the “times 7” column, and see that $8 \cdot 7 = 8$.

**Binary Operations**

Given a set $A$, a binary operation is a function from the ordered pairs of elements from $A$ back to $A$. In particular, a binary operation $\ast$ takes any two elements from $A$, say $a, b \in A$, and associates the pair with a unique element $a \ast b \in A$. In function notation, we might write $f(a, b) = a \ast b$, although we’ll just recognize that $a \ast b$ is the sum, product, etc. of $a$ and $b$.

ordered pairs. Be aware that a binary operation will distinguish between $a \ast b$ and $b \ast a$. These will often be equal, but this is not always the case.

every ordered pair. A binary operation must assign a value in the set to every ordered pair. For example, normal division on the integers is not a binary operation, since some pairs, like $5 \div 3$, don’t have values in $\mathbb{Z}$. On the other hand, division on the positive reals is a binary operation.

a unique element. Just so things make sense, we would like to be able to use $a \ast b$ as the name of an element. In particular, we don’t want an operation to have multiple results. It’s hard to mess this up, since all of our normal terminology takes care to avoid ambiguous results. A contrived example might be something like $a \ast b = \pm \sqrt{a^2 + b^2}$ defined on the reals.

**Integers Modulo $n$**

In clock arithmetic, we were dealing with the set of integers modulo 12. We can do the same thing with any positive integer $n$ to define $\mathbb{Z}_n$.

$\mathbb{Z}_2$. In $\mathbb{Z}_2$, we’re dealing with the remainders after dividing by 2, so $\mathbb{Z}_2 = \{0, 1\}$. The binary operations $+$ and $\cdot$ on $\mathbb{Z}_2$ basically correspond to what we know about adding and multiplying odd and even numbers. For example, $1 + 1 = 0$ corresponds to saying “odd plus odd is even.”
**Divisors of zero.** The study of binary operations lies mainly in the field of *modern algebra* or *abstract algebra*, and in its “purest” form, algebra seeks to understand the significance of the various properties that binary operations might have. You have probably heard of things like the commutative property or the distributive property. Here’s another that you probably haven’t seen before.

In normal high school algebra, you could solve a quadratic equation as follows. Given

\[(21) \quad x^2 - 2x - 3 = 0,\]

you could factor the expression on the left side to get

\[(22) \quad (x - 3)(x + 1) = 0,\]

and then conclude that

\[(23) \quad x = 3, -1.\]

Being able to do this depends on the fact that if \((x - 3)(x + 1) = 0\), then we must have either \(x - 3 = 0\) or \(x + 1 = 0\). It follows that \(x = 3\) or \(x = -1\).

The basic property we’re using is that in the real numbers, if \(ab = 0\), then either \(a = 0\) or \(b = 0\).

So what if we don’t have that property? We can understand that a little better, if we had a binary operation that didn’t. In \(\mathbb{Z}_{12}\) we don’t. For example, \(2 \cdot 6 = 0\) and \(3 \cdot 4 = 0\). In this case, 2, 6, 3, and 4 are *divisors of zero*.

Let’s try to solve the quadratic equation that looks the same as the one above in \(\mathbb{Z}_{12}\). In other words, solve

\[(24) \quad x^2 - 2x - 3 = 0.\]

We could factor this as

\[(25) \quad (x - 3)(x + 1) = 0.\]

Is that right? Try multiplying it out. Are \(x = 3\) and \(x = -1\) solutions? Yes. OK so far. We still have that \(0 \cdot a = 0\) in \(\mathbb{Z}_{12}\).

Something different, however, is that \(2 \cdot 6 = 0\). Note that if \(x = 5\), then we have

\[(26) \quad ((5) - 3)((5) + 1) = 2 \cdot 6 = 0,\]

and \(x = 5\) is also a solution.

**Quiz 07b**

1. Are there any other solutions? I got \(x = 3, 5, 9, 11\).

2. Is the factoring step valid? Certainly \((x - 3)(x + 1) = x^2 - 2x - 3\). Try multiplying out

   a. \((x - 5)(x - 9)\)

   b. \((x - 3)(x - 11)\)

   c. \((x + 7)(x + 3)\)

3. Can you figure out which \(\mathbb{Z}_n\)’s have (non-zero) divisors of zero, and which do not?
OTHER BINARY OPERATIONS

Playing with weird binary operations helps us understand the normal ones better. As an application, we would generally prefer a symbolic calculation than to work with the things they represent. Like if George puts 52 cows in a corral, Sarah puts 37, and Zoe takes 5 out. How many cows in the corral? Symbolically that’s easy, $52 + 37 - 5 = 84$. Going out into the corral and counting is harder.

I’m going to take a while on this next example, because I think everyone should see it.

KNOTS AND THE CONWAY POLYNOMIAL

Topology is the study of continuously deformable objects. If you can bend and stretch one object so that it ends up looking like another (following certain rules), then we might say that those two objects are topologically equivalent.

One area within topology is called knot theory. Roughly, a knot is like a piece of rope sitting in space with the ends attached. A knot can also be multiple loops of rope, which we’ll call links.
In the picture below, we have diagrams for three knots. The first is completely unknotted, and it has an appropriate name, the *unknot*. The second knot is basically two unknotted linked together. It’s called a *simple link*. The third knot is called a *trefoil*.

If we can deform a knot by bending, stretching, or twisting (but not breaking) so that it looks like another, then those two knots are *the same knot*.

In the picture below (ignoring the arrows for now), we can deform one into the unknot, one into a simple link, and the middle one is a new knot.

**Why knots?** Knot theory has been an important area of recent research, I think, because they have been found to code the same information as the possible shapes of our universe, but in a form more palatable to our brains. Also, the ideas used to study knots can extend to other situations, where curves cross. For example, the way DNA tangles affects how it behaves, and it displays knot theory like structures. Just off the top of my head, I would think that having a computer pick roads out of an aerial photograph and choosing routes (as in Google Maps) might be a knot theory type problem.

Mostly, I just think that it’s cool to see how out-of-the-box mathematicians can be.

We’ll finish for today with a question that will frame what we’re going to do.

Is it possible for two knots to cancel each other out?
Let me explain a little more. I’m going to define a binary operation on knots called the **connect sum**. Basically, we’re going to take two knots, cut each knot without disturbing the crossings, and then join them together by the loose ends to form a new knot. For example, the picture below shows the connect sum of two trefoils.

![Connect Sum of Two Trefoils](image)

**Homework 07**

1. Do the following division problems in \( \mathbb{Z}_{12} \) using the multiplication table. You might have more than one possible answer, give all of them. You also might not have any, just say DNE (does not exist) for these.
   
   a. \( 7 \div 5 \).
   
   b. \( 3 \div 6 \).
   
   c. \( 8 \div 10 \).
   
   d. \( 5 \div 2 \).
   
   e. \( 2 \div 5 \).

2. If we’re trying to solve an equation like \( 4x = 6 \), we can’t just divide by 4, because 4 does not have a multiplicative inverse. We have to, like some of the problems in Problem 1, just look at the table, and see which elements of \( \mathbb{Z}_{12} \) satisfy “4 times \( x \) equals 6.” Looking at the table, we see there are no solutions. For the following, find all solutions, if any.
   
   a. \( 5x = 9 \).
   
   b. \( 3x = 0 \).
   
   c. \( 4x = 8 \).
   
   d. \( 11x = 4 \).
   
   e. \( 9x = 3 \).

3. Since every element in \( \mathbb{Z}_{12} \) has an additive inverse, subtracting an element from both sides of an equation is fine. Solve the following by subtracting something from both sides of the equation, and then proceeding as in Problem 2.
   
   a. \( 3x + 7 = 11 \).
   
   b. \( 6x + 4 = 0 \).
   
   c. \( 2x + 10 = 8 \).
   
   d. \( 11x + 5 = 7 \).
   
   e. \( 4x + 3 = 5 \).
OK. So the question becomes: Can the connect sum of two knots be equivalent to the unknot?

4. Clearly, there are two knots that can be connect summed and give you the unknot. What are the two knots (they can be the same two knots)?

5. Is it clear that two mirror-image trefoils can or can’t untie each other?

6. Do you think it’s possible for two knots to untie each other?

Answers on next page.
1) Should look very similar to 4. 1a) 11. b) DNE. c) 2 and 8. d) DNE. e) 10.
2a) \( x = 9 \). b) \( x = 0, 4, 8 \). c) \( x = 2, 5, 8, 11 \). d) \( x = 8 \). e) \( x = 3, 7, 11 \).
3a) \( 3x = 4 \). No solutions.
   b) \( 6x = 8 \). No solutions.
   c) \( 2x = 10 \). \( x = 5, 11 \).
   d) \( 11x = 2 \). \( x = 10 \).
   e) \( 4x = 2 \). No solutions.
4) Two unknots. That’s the boring answer. Are there any interesting ones?
5) I don’t know if it’s clear or not, but it doesn’t seem like they can untie each other.
6) Knots untie on me all the time when I’m trying hold something down, but extension cords and garden hoses get hopelessly tangled. It’s really hard to say.