Objectives: Define the Conway Polynomial.

1. The Conway Polynomial

We can think of knots as coding information in a geometric manner. We’d like to be able to pull that information out in terms of symbols that we can manipulate. Now a knot is a continuous/geometric object, but if you focus on the crossings in a knot diagram, there are only a finite number of these, and how they cross and how they’re connected to each other is really all the information they contain. If we look at them that way, therefore, they are discrete math objects.

Complicating matters is that a particular knot can have many different representations. The differences between these are mainly extraneous crossings that we can unravel mentally in the simpler cases. Again, a symbolic representation that removes this extraneous information would make many things easier.

In 1923, J.W. Alexander found a way of extracting useful information the system of crossings and depositing it in a polynomial. This is an incredibly wild idea that works really well. Knot theorists are still finding new and different ways of attaching polynomials to knots, and this is probably the most fruitful method for solving problems concerning knots. In 1970, John Conway found a surprisingly nice way of associating knot polynomials with knots. Conway is one of the great mathematicians of the 20th century. I didn’t see the movie, but I read *A Beautiful Mind* about John Forbes Nash. At Princeton in the 1930’s, the students and faculty used to hang out in the department lounge and play games. These games led to a lot of significant ideas. Nash and Conway were participants, and I believe people like Shapley, Shubik, Banzhaf, and Arrow were in this crowd too.

This is how you find Conway’s polynomials. First, in addition to knots, we’re going to throw in things called *links*. A link is simply a collection of knots considered together as one object. The Figure below shows two examples of links. One is a pair of unknots, and the other is an unknot together with a trefoil. Links don’t actually have to link.

In addition, we are going to place an orientation on each knot and link, so the Conway polynomials will actually be assigned to *oriented* knots and links.

OK. Here are the rules.
**Rule 1.** To each oriented knot or link $K$, there will be associated a polynomial with integer coefficients $\nabla_K$.

**Rule 2.** If $K$ is the unknot, then $\nabla_K = 1$.

**Rule 3.** If diagrams for three knots differ only at one crossing, where at this crossing, knot $R$ looks like the right-handed crossing shown in the picture below, knot $L$ looks like the left-handed crossing, and knot $S$ looks like the smoothed crossing, then the three corresponding Conway polynomials are related by

\[
\nabla_R - \nabla_L = x \cdot \nabla_S.
\]

This may sound a bit odd, but it is really quite brilliant. An example or two should make this clear.

Consider the knots shown in the picture below. Let’s say we started with an unknot and the diagram on the left. The arrows indicate an orientation on this knot, and if you compare this crossing with the crossings described above, you’ll see that this is a right-handed crossing.

We can change this crossing in two ways. One way is to drop the over-crossing to an under-crossing. This gives us the diagram in the middle, and now the crossing is left-handed. The other way is to remove the crossing altogether. This gives us the diagram on the right. Note that both of these changes leaves the rest of the knot unchanged (including the orientation). The three knots we’ll label as $R$, $L$, and $S$.

\[
\begin{align*}
R & \quad \quad \quad \quad \quad \quad L & \quad \quad \quad \quad \quad \quad S
\end{align*}
\]

Rule 3 tells us that the Conway polynomials for these knots are related by the equation

\[
\nabla_R - \nabla_L = x \cdot \nabla_S.
\]

Both $R$ and $L$ are unknots, so Rule 2 tells us that their Conway polynomials must be $\nabla_R = 1$ and $\nabla_L = 1$. The equation is now

\[
1 - 1 = x \cdot \nabla_S.
\]

It follows that

\[
\nabla_S = \frac{0}{x} = 0.
\]

In other words, the Conway polynomial for a pair of unlinked unknots is 0.

**Basic Principle 1.** It turns out that unlinked links of any kind will have Conway polynomial equal to 0.
In this picture, we have three more knot diagrams differing at one crossing. This crossing is right-handed on the first, left-handed on the middle one, and smoothed on the right. Here $R$ is an unlinked link, $L$ is a linked link, and $S$ is an unknot. We know two of these, so the equation from Rule 3 tells us

\[ 0 - \nabla_L = x \cdot 1. \]

It follows that $\nabla_L = -x$. This particular knot is called a *simple link*. There are actually two, a left-handed one, and a right-handed one. This one is left-handed.

2. Quiz 08

OK. You guys do one. The middle knot in this picture is a trefoil. Find its Conway Polynomial.

3. More on the Conway Polynomial

So we’ve found the Conway Polynomial for a few knots. If we want, we could just continue randomly, and build up a catalog of Conway Polynomials. That’s not the most efficient method, especially if you have a particular knot in mind.

**Basic Principle 2.** *It’s better to just start with the knot you’re interested in. If your knot is in a relatively simple form (no extraneous crossings), then changing any of the crossings should make it “unravel.” If you keep doing that, you’ll eventually end up with unknots and unlinks. We’re already designating unknots to have Conway Polynomials equal to 1, and I’ve told you that all unlinked links have Conway Polynomial equal to 0.*

Let me illustrate with the figure-eight knot.
The figure-eight knot is the knot $B$ in the picture above, and I’ve given it an orientation as shown by the arrows (a different orientation might yield a different polynomial). I’ve also chosen to work with the crossing in the middle, but any of the crossings should work equally well. With the orientation I’ve chosen, this crossing is left-handed. Changing the crossing yields a knot with a right-handed crossing, the knot $A$, which is an unknot, and a knot with a smoothed crossing, the knot $C$, which is a simple link.

The one we don’t know is $C$, so let’s focus on it. For no particular reason, I’ll work with the crossing on the upper right, which is right handed, so I’ll put it first in the picture below.

Changing the crossing to a left-handed crossing and a smoothed crossing gives us the knots $D$ and $E$, which are an unlinked link and an unknot, and we know those. This gives us the following.

$$\nabla_A - \nabla_B = x \cdot \nabla_C$$

$$\nabla_C - \nabla_D = x \cdot \nabla_E$$

to get

$$1 - \nabla_B = x \cdot \nabla_C$$

$$\nabla_C - 0 = x \cdot 1.$$  

The second equation tells us that $\nabla_C = x$, and it follows that

$$\nabla_B = 1 - x^2,$$

so this is the Conway polynomial for the figure-eight knot.

4. Connect Sums correspond to Polynomial Multiplication

One really amazing property of the Conway polynomial is the following.

$$\nabla_A \# B = \nabla_A \cdot \nabla_B.$$  

For example, if we were to take the connect sum of a figure-eight knot and a trefoil, then the Conway polynomial for the connect sum would be

$$(1 - x^2)(x^2 + 1) = 1 - x^4.$$
This by itself proves that the figure-eight and trefoil cannot be connect sum inverses, since the connect sum in this case is not the unknot, which would have Conway polynomial 1.

Let’s go through the computation of the Conway polynomial for the connect sum of a figure-eight and a trefoil. In the process, we’ll see why the Conway polynomials multiply.

Look at picture above. I’ve connect summed the trefoil to the figure-eight. I made the trefoil small, because I’m not going to do anything to it. I’m just going to repeat the computations for the figure-eight knot that we just did. The connect sum of the trefoil and figure-eight knot is knot $B$, and it has a left-handed crossing where the arrows are. Knots $A$ and $C$ have the right-handed and smoothed versions.

$\nabla_A - \nabla_B = x \cdot \nabla_C$

Continuing on as we did before, we break $C$ down further. The upper-right crossing is right-handed, so if we change that crossing to its left-handed and smoothed versions, we get the knots below.

$\nabla_C - \nabla_D = x \cdot \nabla_E$

We get the two equations

$\nabla_A - \nabla_B = x \cdot \nabla_C$

$\nabla_C - \nabla_D = x \cdot \nabla_E$,

which look the same as with the figure-eight by itself, but the knots $A$ and $E$ are trefoils instead of unknots. The knot $D$ is still an unlink. Let’s let $\nabla_T$ be the Conway polynomial for the trefoil, and substituting this
information into the equations we got before with the plain figure-eight knot, we get
\[ \nabla_T - \nabla_B = x \cdot \nabla_C, \]
\[ \nabla_C - 0 = x \cdot \nabla_T. \]

This gives us
\[ \nabla_C = x \cdot \nabla_T, \]
and then
\[ \nabla_T - \nabla_B = x \cdot (x \cdot \nabla_T). \]

Therefore,
\[ -\nabla_B = x^2 \cdot \nabla_T - \nabla_T = (x^2 - 1) \cdot \nabla_T, \]
and
\[ \nabla_B = -(x^2 - 1) \cdot \nabla_T = (1 - x^2) \cdot \nabla_T. \]

In this particular case, if we let \( F \) be the figure-eight knot, then we’ve just established that
\[ \nabla_{F\#T} = \nabla_F \cdot \nabla_T. \]

5. Homework 08

1. I told you that any unlinked link has Conway Polynomial equal to 0. You’re going to demonstrate this. Suppose we have an unlinked link where the two parts of the link are the knots \( A \) and \( B \). You want to show that the Conway Polynomial must be 0. Consider the funny connect sum of \( A \) and \( B \) shown below.

   ![Mystery Knot A and Mystery Knot B](image)

   a. Is the crossing shown left-handed or right-handed?

   b. Let’s say that \( A\#B = R \). The other two knots/links will be \( L \) and \( S \), if we change this crossing. How are \( L \) and \( R \) related?

   c. So how are \( \nabla_L \) and \( \nabla_R \) related?

   d. That would mean that \( \nabla_R - \nabla_L =? \)

   e. So \( S \) could be any unlinked link. What is \( \nabla_S =? \)
2. Consider the knot shown below. Find the Conway Polynomial.

Here’s a worksheet for you to use. You might want to trace it onto another piece of paper.

Answers: 1a) Right handed. b) They are the same knot. c) They are equal. d) Equals 0. e) 0.
2 I got $2x^2 + 1$. 