You may be familiar with taking the union and intersection of two sets. I would like to point out that union and intersection are binary operations acting on the subsets of some larger set, and so we can talk about an algebra of sets. In the following, $A \cap B$ is the intersection, $A \cup B$ is the union, and $A^c$ is the complement (the things not in $A$).

**Quiz 09A**

We’ll start off with a problem to get us warmed up. With the sets $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, and $B = \{4, 5, 6\}$, simplify

$$ (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B^c). $$

**Boolean Algebras**

Especially for a more complicated example, if we could simplify this like we would an algebraic expression, you could see that that might be a good alternative. In fact, the above expression is equivalent to $A \cup B^c$ for any sets $A$ and $B$, and we could establish that with a set algebra.

Let me start off by reviewing the concepts of union, intersection, and set complement. We will always be talking about subsets of some universal set, $U$. For example, we’re talking about the multiples of 4, the universal set would be the integers, and the multiples of 4 would be a subset of the integers. The universal set will be represented by the box in the Venn diagrams below.

**Intersection.** The intersection of two sets $A$ and $B$ is the set consisting of all the elements that belong to both $A$ and $B$, as shown in the Venn diagram below.
Union. The union of two sets $A$ and $B$ is the set consisting of those elements of $U$ that belong to at least one of $A$ and $B$, as shown in the Venn diagram below.

![Venn Diagram for Union](image)

Complement. The complement of a set $A$ is the set consisting of those elements of $U$ that do not belong to $A$, as shown below.

![Venn Diagram for Complement](image)

Standard Axioms for a Boolean algebra. A Boolean algebra is a set $X$ with two binary operations $\lor$ and $\land$ and a unary operation $\sim$ defined on $X$. There are two special elements 0 and 1, and along with any other elements $a$, $b$, and $c$ in $X$, we have

\[
\begin{align*}
    a \lor (b \lor c) &= (a \lor b) \lor c & a \land (b \land c) &= (a \land b) \land c \\
    a \lor b &= b \lor a & a \land b &= b \land a \\
    a \lor 0 &= a & a \land 1 &= a \\
    a \lor (b \land c) &= (a \lor b) \land (a \lor c) & a \land (b \lor c) &= (a \land b) \lor (a \land c) \\
    a \lor \sim a &= 1 & a \land \sim a &= 0
\end{align*}
\]

The axioms of a Boolean algebra correspond to basic properties of union, intersection, and complement acting on sets. In fact, any property of sets involving union, intersection, and complement can be proven from these basic properties.

Corresponding set properties. The following are the corresponding set properties.

Associative Laws. The associative laws basically say that in expressions involving only unions or only intersections, the parentheses don’t matter. In particular, the union, $A \lor B \lor C$, is the set of elements that belong to at least one of the three sets.

\[
A \lor (B \lor C) = (A \lor B) \lor C \quad \text{and} \quad A \land (B \land C) = (A \land B) \land C
\]

Commutative Laws. The order does not matter in a union or intersection.

\[
A \lor B = B \lor A \quad \text{and} \quad A \land B = B \land A
\]
Existence of Identities. The 0 in a generic Boolean algebra goes with the empty set $\emptyset = \{ \}$. The 1 goes with the universal set $U$.

\[(5) \quad A \cup \emptyset = A \quad \text{and} \quad A \cap U = A\]

Distributive Laws. These are probably the least obvious of the basic axioms. We looked at Venn diagrams in class.

\[(6) \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and} \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\]

Complementation Laws.

\[(7) \quad A \cup A^c = U \quad \text{and} \quad A \cap A^c = \emptyset\]

Quiz 09B

Draw Venn diagrams that prove the distributive laws.

Some other basic properties. Here are some other obvious set properties beyond the basic Boolean algebra axioms.

\[(8) \quad A \cap \emptyset = \emptyset \]
\[(8) \quad A \cup U = U \]
\[(8) \quad A \cap (A \cup B) = A \]
\[(8) \quad A \cup (A \cap B) = A \]
\[(8) \quad A \cap A = A \]
\[(8) \quad A \cup A = A \]

If $A \cup B = U$ and $A \cap B = \emptyset$, then $B = A^c$.

\[(8) \quad (A \cap B)^c = A^c \cup B^c \]
\[(8) \quad (A \cup B)^c = A^c \cap B^c \]
\[(8) \quad (A^c)^c = A \]

Quiz 09C

Use the properties above to simplify $A \cup (A^c \cap B)$.

Homework 09

Use the properties to simplify $(A \cup B) \cap (A \cup B^c) \cap (A^c \cup B^c)$.

Homework answer: You should get $A \cap B^c$. 