Objectives: Introduce measure, probability, and cardinality.

Earlier in the semester, we briefly discussed cardinality. I’d like to go into this in a bit more detail now. In particular, I want to look at a few ways mathematicians have used to describe how big a particular set is, or isn’t.

The basic concepts of set bigness are ones that you’re already familiar with, counting, length, area, and volume. Counting might seem to be a bit different from the other three, but I like to think of all four together. Length, area, and volume are 1-, 2-, and 3-dimensional measures, and I think of counting as a 0-dimensional measure. This makes sense on several levels. One simple way is that the lower dimensional measures are small compared to the higher dimensional ones. For example, the set of real numbers between 0 and 2, the interval \([0, 2]\) has length 2. It’s area, however, must be zero. There are three integers in the set \([0, 2], \{0, 1, 2\}\), and we would think of one of these as being infinitely smaller than the other. At least in terms of “substantiveness” of the two sets.

A very closely related concept to these measures is that of probability. For example, we could roll a die, and ask what the probability of getting a 1 or 2 is. We’re comfortable reasoning that there are six possible outcomes when rolling a die, and all six are equally likely. Therefore, the probability of getting a 1 or 2 must be

\[ \frac{2}{6} = \frac{1}{3}. \]

Underneath it all, we’re comparing the size of the sets \(\{1, 2\}\) and \(\{1, 2, 3, 4, 5, 6\}\).

Slightly more involved, we might pull a number randomly from the set \([0, 2]\), and ask for the probability that that number lies in the interval \([0, 1]\). Since the lengths of these two intervals are 2 and 1, we would conclude that this probability must be

\[ \frac{1}{2}. \]

Again, we’re comparing the size of the two sets. Note that the probability of the number lying in the open interval \((0, 1)\) must also be \(\frac{1}{2}\), and we are probably comfortable with the idea that a few isolated points won’t change the probability. How many points would matter?

OK. I would like to push these concepts a bit. Suppose still that we’re pulling a number randomly from the interval \([0, 2]\). What is the probability that that number is rational? Irrational? The root of any polynomial? I claim that the probabilities are 0, 1, and 0.

**Cardinality**

As you may recall, cardinality is a concept originating in the mind of Georg Cantor. We will say that two sets have the same cardinality, if they can be put into a one to one correspondence. For example, the sets \(\{1, 2, 3, 4\}\) and \(\{2, 6, 9, 23\}\) have the same cardinality, because

\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
2 & 6 & 9 & 23 \\
\end{array} \]

The concept of cardinality can be thought of as being a generalization of counting, since we can say that a set has \(n\) elements, if it has the same cardinality as \(\{1, 2, 3, \ldots, n\}\).

Recall that the set \(\mathbb{N} = \{1, 2, 3, 4, \ldots\}\) is called the natural numbers. I’ll use the notation \(2\mathbb{N} = \{2, 4, 6, \ldots\}\) to denote the even natural numbers. I’ll also use the notation \(\|A\|\) to denote the cardinality of the set \(A\).
When we talked about this before, we saw that \( \|N\| = \|2N\| \), since
\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & \ldots \\
\end{array}
\]
We saw also that since
\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & -1 & 1 & -2 & 2 & -3 & 3 & \ldots \\
\end{array}
\]
that \( \|N\| = \|Z\| \).

**Countably Infinite.** Any set with the same cardinality as \( \|N\| \) is said to be **countably infinite**. Any set that is countably infinite or finite is said to be **countable**.

**Convenient Facts about Cardinality.** Before we go on, it will be convenient to state some facts that seem obvious, but are awkward to prove (and so we won’t).

**Basic Principle 1.** If \( A, B, \) and \( C \) are sets, \( B \subset C \), and \( \|A\| = \|B\| \), then the cardinality of \( A \) is smaller or equal to the cardinality of \( C \). I’ll write \( \|A\| \leq \|C\| \).

Note that we consider \( A \subset A \). I probably should have defined **smaller cardinality**. I’ll do that now. \( \|A\| < \|B\| \), if there is no one-to-one correspondence between \( B \) and any subset of \( A \).

**Basic Principle 2.** If \( \|A\| \leq \|B\| \) and \( \|B\| \leq \|A\| \), then \( \|A\| = \|B\| \).

**RATIONALS ARE COUNTABLE**

Note that a set is countable if it can be listed out in a sequence. That’ll save me typing in those arrows. Let me start with the positive rationals, which I’ll call \( \mathbb{Q}^+ \). The positive rationals can be put in an infinite array such that each row consists of all fractions with the same denominator as
\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots \\
5 & 6 & 7 & 8 & 9 & 10 & 11 & \ldots \\
6 & 7 & 8 & 9 & 10 & 11 & 12 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

We can squeeze this array into a sequence by taking diagonals from lower left to upper right. In other words, by listing out the fractions whose numerator and denominator add up to 2, then 3, then 4, etc.
\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & \ldots \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \ldots \\
\end{array}
\]

Now we are listing each rational number multiple times, so in essence, we have established a one-to-one correspondence between \( \mathbb{Q}^+ \) and a subset of \( \mathbb{N} \). In other words, \( \|\mathbb{Q}^+\| \leq \|\mathbb{N}\| \). Since \( \mathbb{N} \subset \mathbb{Q}^+ \), however, we must also have \( \|\mathbb{N}\| \leq \|\mathbb{Q}^+\| \), and so \( \mathbb{Q}^+ \) is countable.

Since \( \mathbb{Q}^+ \) is countable, we can put it into a sequence \( \{ q_0, q_1, q_2, q_3, \ldots \} \), and so all the rationals can be listed as
\[
0, -q_0, q_0, -q_1, q_1, -q_2, q_2, \ldots
\]
REALS ARE NOT COUNTABLE

I’ll just give you the basic idea behind the Cantor diagonalization process, which establishes that the real numbers are not countable. It goes something like this.

We are going to show that the set \((0, 1)\), the open interval consisting of all real numbers between 0 and 1, is not countable. We will assume that \((0, 1)\) is countable, and then arrive at a contradiction. If \((0, 1)\) is countable, then we can put all of the numbers in \((0, 1)\) in a sequence \(x_1, x_2, x_3, \text{etc.}\) Each of these numbers has a decimal representation, which is possibly infinite. Just to see how the argument goes, suppose the sequence looks like the following.

\[
\begin{align*}
  x_1 &= 0.2679282834736\ldots \\
  x_2 &= 0.757575757575\ldots \\
  x_3 &= 0.500000000000\ldots \\
  x_4 &= 0.847048470394\ldots \\
  x_5 &= 0.9925249697979\ldots \\
  &\vdots
\end{align*}
\]

(9)

Note that I’ve highlighted the 1st digit in \(x_1\), the 2nd digit in \(x_2\), and in general, the \(n\)th digit in \(x_n\). The contradiction will be the existence of a real number between 0 and 1 that does not appear on this list. I can construct such a number by choosing digits that differ from the highlighted digits. For example, my number \(y\) will have its \(n\)th digit different from the \(n\)th digit in \(x_n\). These digits are 5, 5, 0, 6, 2, etc. The digits of \(y\) have to be different from these. I could let

\[
y = 0.44444\ldots
\]

(10)
or anything that differs from every \(x_i\) in the list. The conclusion is that \((0, 1)\) is not countable.

**Homework 11**

Think about this.