Objectives: Example of an application of cardinality to continuous stuff.

Whenever you’re encountering a new concept, I think it’s a good idea to kind of step over to the other side, and think about things you’re not going to do. It’s good for your brain to know where the boundaries are. In this case, discrete stuff is pretty much always countable. So what’s different about uncountable? By the way, sets that have bigger cardinality than \( \mathbb{N} \) are said to be uncountable.

First off, I’m hoping that you are wondering if there are larger cardinalities than \( \| (0, 1) \| \). Very quickly, note that the tangent function gives us a one-to-one correspondence between the interval \( (-\frac{\pi}{2}, \frac{\pi}{2}) \) to \( \mathbb{R} \). The picture below shows a Maple graph of the tangent function.

![Maple graph of the tangent function](image)

Playing with functions like this, you can see that any interval, including \( (-\infty, \infty) \), has the same cardinality.

Given any set \( A \), the power set of \( A \) (denoted \( P(A) \)) is the set consisting of all of the subsets of \( A \). It turns out that \( \| P(A) \| \) is always larger than \( \| A \| \). A generalization of Cantor’s diagonalization argument establishes this. To get a feeling for how this works, let’s do the proof on \( \mathbb{N} \) and \( P(\mathbb{N}) \).

Let us suppose that there is a one-to-one correspondence between \( \mathbb{N} \) and \( P(\mathbb{N}) \). This one-to-one correspondence is a function \( f : \mathbb{N} \to P(\mathbb{N}) \), and \( f(n) \) for any natural number \( n \) is a subset of \( \mathbb{N} \). It makes sense, therefore, to ask if \( n \in f(n) \). We’re going to be interested in all those \( n \)’s such that \( n \notin f(n) \). Consider the following sample correspondence.

\[
\begin{array}{c|c|c}
\mathbb{N} & f & P(\mathbb{N}) \\hline
1 & \{2, 3\} & 1 \\
2 & \{2, 4, 6, \ldots\} & \\
3 & \{1, 3, 5, \ldots\} & \\
4 & \{1, 10\} & 4 \\
5 & \{5, 10, 11\} & \\
6 & \{5, 6, 7, 8, \ldots\} & \\
7 & \{8\} & 7 \\
\vdots & \vdots & \\
\end{array}
\]

(1)

OK. We’re assuming that every subset of \( \mathbb{N} \) is in the table somewhere. We’re going to prove that this can’t be true by coming up with a subset of \( \mathbb{N} \) that isn’t in the table. The last column lists the elements of \( \mathbb{N} \) that satisfy \( n \notin f(n) \). Let

\[
A = \{ n \mid n \notin f(n) \} = \{ 1, 4, 7, \ldots \}.
\]

The set \( A \) is not in the table. If it were, then \( A = f(n) \) for some \( n \). Is \( n \in A \)? If so, then \( n \notin A \). Is \( n \notin A \)? If so, then \( n \in A \). Oops.
Note that putting the subsets in a table like this is not critical to the argument, and so the argument can be used on any power set. For example, it can be used to show that \( \| \mathbb{R} \| < \| P(\mathbb{R}) \| \). Cantor is a total freak.

**Lebesgue measure**

Henri Lebesgue (most people pronounce this like: luh BAYg) is famous mostly because his name is associated with the *Lebesgue integral* and the closely related *Lebesgue measure*. The Lebesgue integral is mathematically superior to the Riemann integral you saw in Calculus II, but Riemann is way more famous, because he did a lot more well-known stuff.

Lebesgue measure generalizes the concept of the length of an interval. The length of the interval \([0, 2)\) is 2, and that is also its Lebesgue measure, which we’ll denote as

\[ \mu([0, 2)) = 2. \]

There are other kinds of subsets of \(\mathbb{R}\) other than intervals. For example, it makes sense to say that

\[ \mu([-1, 1] \cup (3, 4) \cup \{7\}) = 2 + 1 + 0 = 3. \]

So what’s Lebesgue’s big idea? He extended this idea of length to more complicated sets like \(\mathbb{Q}\). So suppose we have a set \(A\). If \(A\) is a subset of a collection of open intervals

\[ A \subset (a, b) \cup (c, d) \cup (e, f), \]

for example, then

\[ \mu(A) \leq \mu(a, b) + \mu(c, d) + \mu(e, f) = (b - a) + (d - c) + (f - e). \]

A collection of intervals that contain \(A\) is called a **cover**. Now the measure of any cover makes sense, and the smallest possible total measure we get this way should be equal to the measure of \(A\). If there isn’t a single smallest cover, Lebesgue just takes a limit. In particular, \(\mu(A)\) is the greatest lower bound of the measures of all possible countable covers of \(A\).

As a simple example consider a single point \(\{0\}\). We can cover this set with a single open interval \((-\frac{1}{n}, \frac{1}{n})\), which has length \(\frac{2}{n}\). The greatest lower bound of these numbers \(1, \frac{1}{2}, \frac{1}{3}, \) etc. is 0.

*Convenient fact.*

\[ \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1. \]

**Quiz 12**

1. \(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots =\)
2. \(1 + \frac{1}{4} + \frac{1}{8} + \cdots =\)
3. \(3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots =\)
4. \(\epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \cdots =\)

\[ \mu(\mathbb{Q}) =? \]

OK. So what is \(\mu(\mathbb{Q})\)? It is zero. How do we know? First, \(\mathbb{Q}\) is countable, so we can list it out as a sequence

\[ \mathbb{Q} = \{ q_1, q_2, q_3, \ldots \}. \]
We can put each rational number inside an open interval,

\[ q_1 \in (q_1 - \frac{\epsilon}{2}, q_1 + \frac{\epsilon}{2}) \]
\[ q_2 \in (q_2 - \frac{\epsilon}{4}, q_2 + \frac{\epsilon}{4}) \]
\[ q_3 \in (q_3 - \frac{\epsilon}{8}, q_3 + \frac{\epsilon}{8}) \]
\[ q_4 \in (q_4 - \frac{\epsilon}{16}, q_4 + \frac{\epsilon}{16}) \]
\[ \vdots \]

and in general,

\[ q_i \in \left( q_i - \frac{\epsilon}{2^i}, q_i + \frac{\epsilon}{2^i} \right) \]

for any positive number \( \epsilon \). Each interval has length (i.e. Lebesgue measure) \( 2 \cdot \frac{\epsilon}{2^i} \), and so

\[ \mu(\mathbb{Q}) \leq \sum_{i=1}^{\infty} 2 \cdot \frac{\epsilon}{2^i} = 2\epsilon. \]

This is true for any number \( \epsilon > 0 \), so \( \mu(\mathbb{Q}) = 0 \).

It follows that \( \mu(\mathbb{R}) = \infty, \mu(\mathbb{I}) = \infty, \mu(\mathbb{I} \cap (0,1)) = 1, \) etc.

**Homework 12 - The Cantor set**

I promise to move on to more boring things, but here’s one last thing. Are there uncountable sets with measure zero?

Consider the following sets. The first is boring,

\[ C_0 = [0,1]. \]

The next set is obtained by removing the middle third,

\[ C_1 = \left[ 0, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, 1 \right]. \]

Then we remove the middle thirds of these two intervals

\[ C_2 = \left[ 0, \frac{1}{9} \right] \cup \left[ \frac{2}{9}, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, \frac{7}{9} \right] \cup \left[ \frac{8}{9}, 1 \right]. \]

1. What is \( C_3 \)?
2. What is \( \mu(C_0) \)? \( \mu(C_1) \)? \( \mu(C_2) \)? \( \mu(C_3) \)?
3. What is \( \mu(C_i) \)?

The Cantor set. So we have this countable sequence of sets, and each is a subset of the one before (i.e., \( C_{i+1} \subset C_i \)). The Cantor set, \( C \), is the limit of this sequence. In particular, \( C \) is the intersection of all the \( C_i \)'s.

4. What is \( \mu(C) \)?

Is there anything in the Cantor set? Note that once a number is an endpoint in one of the \( C_i \)'s, it’s an endpoint for all the rest. Therefore, each endpoint must be in the intersection of all the \( C_i \)'s, and so is in \( C \).

Is there anything else in \( C \)? Here’s one way to see. We’ve talked about Base 2 numbers. Remember? We can also talk about Base 3 numbers. For example, in Base 3,

\[ 2101_3 = 2 \cdot 3^3 + 1 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 = 54 + 9 + 0 + 1 = 64. \]

5. What is \( 212_3 \) equal to? (In Base 10).
In Base 10, we carry things through to negative powers of 10. For example,
\begin{equation}
12.56 = 1 \cdot 10^1 + 2 \cdot 10^0 + 5 \cdot 10^{-1} + 6 \cdot 10^{-2}.
\end{equation}
We can do the same thing in Base 3. For example,
\begin{equation}
11.212_3 = 1 \cdot 3^1 + 1 \cdot 3^0 + 2 \cdot 3^{-1} + 1 \cdot 3^{-2} + 2 \cdot 3^{-3}
\end{equation}

6. Look at $C_1$. The endpoints, expressed in Base 3, are 0.03, 0.13, 0.23, and 1.03. Which of the following two “trigit” numbers are in $C_1$: 0.003, 0.013, 0.023, 0.103, 0.113, 0.123, 0.203, 0.213, 0.223, 1.003?

7. Now look at $C_2$. Which three trigit numbers are in $C_2$?

8. Do you see the pattern? If not, look at the four trigit numbers that are in $C_3$.

9. It turns out that $0.22222\ldots_3 = 1.03$. In other words, infinitely repeating 2’s in Base 3 give you 1.03. This means, for example, that $0.201_3 = 0.2002222\ldots_3$. Your pattern from Problem 8, therefore, can be restated as, “the Cantor set consists of those Base 3 numbers that use only 0’s and 2’s. Can you explain why $C$ is uncountable from that?
Consider the following sets. The first is boring,
\((18)\quad C_0 = [0, 1].\)
The next set is obtained by removing the middle third,
\[(19)\quad C_1 = \left[ 0, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, 1 \right].\]
Then we remove the middle thirds of these two intervals
\[(20)\quad C_2 = \left[ 0, \frac{1}{9} \right] \cup \left[ \frac{2}{9}, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, \frac{7}{9} \right] \cup \left[ \frac{8}{9}, 1 \right].\]

1. **What is** \(C_3?\)
   \(C_3 = \left[ 0, \frac{1}{27} \right] \cup \left[ \frac{2}{27}, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, \frac{7}{27} \right] \cup \left[ \frac{8}{27}, \frac{1}{9} \right] \cup \left[ \frac{2}{9}, \frac{19}{27} \right] \cup \left[ \frac{20}{27}, \frac{7}{9} \right] \cup \left[ \frac{8}{9}, \frac{26}{27} \right] \cup \left[ \frac{26}{27}, 1 \right].\)

2. **What is** \(\mu(C_0)? \mu(C_1)? \mu(C_2)? \mu(C_3)?\)
   \(\mu(C_0) = 1. \mu(C_1) = 1 - \frac{1}{3} = \frac{2}{3}. \mu(C_2) = \frac{2}{3} - 2 \cdot \frac{1}{9} = \frac{4}{9}. \mu(C_3) = \frac{4}{9} - 4 \cdot \frac{1}{27} = \frac{8}{27}.\)

3. **What is** \(\mu(C_i)?\)
   \(\mu(C_i) = \frac{2^i}{3^i}.\)

The **Cantor set**. So we have this countable sequence of sets, and each is a subset of the one before (i.e., \(C_{i+1} \subset C_i\)). The **Cantor set**, \(C\), is the limit of this sequence. In particular, \(C\) is the intersection of all the \(C_i\)’s.

4. **What is** \(\mu(C)?\)
   \(\mu(C) = \lim_{i \to \infty} \frac{2^i}{3^i} = \lim_{i \to \infty} \left( \frac{2}{3} \right)^i = 0.\) Note: The powers of any positive number less than 1 go to 0.

**Is there anything in the Cantor set?** Note that once a number is an endpoint in one of the \(C_i\)’s, it’s an endpoint for all the rest. Therefore, each endpoint must be in the intersection of all the \(C_i\)’s, and so is in \(C\).

Is there anything else in \(C\)? Here’s one way to see. We’ve talked about Base 2 numbers. Remember? We can also talk about Base 3 numbers. For example, in Base 3,
\[(21)\quad 2101_3 = 2 \cdot 3^3 + 1 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 = 54 + 9 + 0 + 1 = 64.\]

5. **What is** \(212_3\) **equal to?** (In Base 10).
   \(212_3 = 2 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0 = 18 + 3 + 2 = 23.\)
In Base 10, we carry things through to negative powers of 10. For example,
\[(22)\quad 12.56 = 1 \cdot 10^1 + 2 \cdot 10^0 + 5 \cdot 10^{-1} + 6 \cdot 10^{-2}.\]
We can do the same thing in Base 3. For example,
\[(23)\quad 11.212_3 = 1 \cdot 3^1 + 1 \cdot 3^0 + 2 \cdot 3^{-1} + 1 \cdot 3^{-2} + 2 \cdot 3^{-3}\]

6. **Look at** \(C_1\). The endpoints, expressed in Base 3, are 0.03, 0.13, 0.23, and 1.03. Which of the following two “trigit” numbers are in \(C_1\): 0.003, 0.013, 0.023, 0.103, 0.113, 0.123, 0.203, 0.213, 0.223, 1.003?\)
   \(0.003 = 0\) (endpoint), \(0.013 = \frac{1}{9}\) (inside), \(0.023 = \frac{7}{27}\) (inside), \(0.103 = \frac{1}{3}\) (endpoint), \(0.113 = \frac{4}{9}\) (out), \(0.123 = \frac{5}{9}\) (out), \(0.203 = \frac{8}{27}\) (endpoint), \(0.213 = \frac{7}{9}\) (inside), \(0.223 = \frac{5}{9}\) (inside), \(1.003 = 1\) (endpoint).

7. **Now look at** \(C_2\). Which three trigit numbers are in \(C_2\)?
I got: 0.000₃, 0.001₃, 0.010₃, 0.020₃, 0.021₃, 0.022₃, 0.100₃, 0.200₃, 0.201₃, 0.202₃, 0.220₃, 0.221₃, 0.222₃, 1.000₃

8. Do you see the pattern? If not, look at the four trigit numbers that are in \( C_3 \).
   The rule seems to be the last non-zero trigit can be a 1, but otherwise, you can only have 0’s and 2’s.

9. It turns out that 0.222222\ldots_3 = 1.0_3. In other words, infinitely repeating 2’s in Base 3 give you 1.0₃. This means, for example, that 0.201₃ = 0.2002222\ldots_3. Your pattern from Problem 8, therefore, can be restated as, “the Cantor set consists of those Base 3 numbers that use only 0’s and 2’s. Can you explain why \( C \) is uncountable from that?

The Cantor set Base 3 expansions use only 0’s and 2’s. There’s a natural one-to-one correspondence with the Base 2 expansions, just change the 2’s to 1’s. These are all the real numbers in \((0, 1)\) expressed in Base 2 (a.k.a. binary). The Cantor set has the same cardinality as \((0, 1)\).