We should occasionally remind ourselves that an infinite decimal is defined in terms of a limit. The standard definition would say that

\[ 0.999\ldots = \sum_{i=1}^{\infty} \frac{9}{10^i} \]

From Calculus II, we know for \( r < 1 \) that a geometric series converges to

\[ \sum_{i=0}^{\infty} ar^i = \frac{a}{1-r} \]

This tells us that

\[ 0.999\ldots = \sum_{i=1}^{\infty} \frac{9}{10^i} = -9 + \sum_{i=0}^{\infty} \frac{9}{10^i} = -9 + \frac{9}{1 - \frac{1}{10}} = -9 + \frac{90}{9} = 1. \]

**Finite cardinality of power sets**

Given a set \( A \), we saw that the power set, \( P(A) \), always has larger cardinality than \( A \). If \( A \) is finite, we can say how many elements \( P(A) \) has.

As a basic fact, we can say that

\[ \| P(A) \| = 2^{|A|} \]

In fact, mathematicians will extend this formula to levels of infinity, for example, \( \| \mathbb{R} \| = 2^{\|\mathbb{N}\|} \).

At least as important as the facts, I think, are the connections with other facts and the underlying structure. So let’s take a look at this.

**Correspondence with Binary Numbers.** Take, for example, the set \( A = \{1, 2, 3, 4\} \). The power set \( P(A) \) is the collection of all the subsets of \( A \), and we’d like to count how many subsets there are. We can describe each subset as a 4-term sequence of 0’s and 1’s. For example,

\[ 1, 0, 1, 1 \leftrightarrow \{1, 3, 4\} \]

and

\[ 0, 1, 1, 0 \leftrightarrow \{2, 3\} . \]

In particular, the first term of the sequence tells us whether the first element of the set is in the subset or not, and so on.

Once we have a 4-term sequence of 0’s and 1’s, we can pair these easily with all the 4-digit (4-bigit, might be more appropriate) binary numbers. Since \( 1111_2 = 8 + 4 + 2 + 1 = 15 \), there must be 16 subsets of \( A \) (since we’re also counting 0000_2). There are 32 5-digit binary numbers, 64 6-digit binary numbers, etc., so this agrees with the original fact.
1. We could define the number 0.9999\ldots as the limit of the sequence
   \begin{align*}
   &0, 0.9, 0.99, 0.999, 0.9999, \ldots .
   \end{align*}
   These numbers are clearly getting bigger and closer to 1, but do they get all the way to 1? This would be equivalent to the following sequence converging to 0:
   \begin{align*}
   &1 - 0.9, 1 - 0.99, 1 - 0.999, \ldots .
   \end{align*}
   a. Simplify the terms of this last sequence.
   b. What is \( \lim_{i \to \infty} \frac{1}{10^i} \)?

2. Recall that for \( r < 1 \), a geometric series converges to
   \begin{align*}
   &\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}.
   \end{align*}
   a. In Base 4, the expansion
   \begin{align*}
   &0.33333\ldots_4 = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots = \sum_{i=1}^{\infty} \frac{3}{4^i}.
   \end{align*}
   Use the geometric series formula to find this sum. Don’t forget that this series starts at \( i = 1 \), not \( i = 0 \).
   b. Do the same thing for 0.0222222\ldots_3 in Base 3.

3. Write the following Base 3 numbers in Base 10.
   a. 12.22_3
   b. 0.011_3
   c. 22_3

4. Write the following numbers in Base 3.
   a. \( \frac{2}{27} \).
   b. \( \frac{22}{27} \).
   c. \( \frac{42}{9} \).

5. Let \( A \) be the set of letters in the alphabet. How many elements are there in \( P(A) \)?

Answers: 1a) 0.1, 0.01, 0.001,\ldots  b) 0.
2a) 0.333\ldots = -3 + 3 \cdot \frac{1}{1-\frac{1}{3}} = -3 + \frac{12}{3} = 1.  b) \frac{1}{3}.
3a) 3 + 2 + \frac{2}{3} + \frac{2}{9} = 11 + \frac{8}{9} = 5.88888\ldots 
   b) \frac{1}{3} + \frac{1}{27} = \frac{1}{27} = 0.148148148\ldots 
   c) 6 + 2 = 8.
4a) 0.0023;  b) \frac{22}{27} = \frac{18+4+1}{27} = 0.2113;  c) \frac{42}{9} = \frac{27+9+6+0}{9} = 11.20_3.
5) \( P(\text{Alphabet}) = 2^{26} = 67,108,864 \).