Objectives: Basic counting formulas
Cantor set problem from the Homework.

\[ 0.9999\ldots = 1 \]

We should occasionally remind ourselves that an infinite decimal is defined in terms of a limit. The standard definition would say that

\[ 0.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = \sum_{i=1}^{\infty} \frac{9}{10^i} \]

From Calculus II, we know for \( r < 1 \) that a geometric series converges to

\[ \sum_{i=0}^{\infty} ar^i = \frac{a}{1-r} \]

This tells us that

\[ 0.9999\ldots = \sum_{i=1}^{\infty} \frac{9}{10^i} = -9 + \sum_{i=0}^{\infty} \frac{9}{10^i} = -9 + \frac{9}{1 - \frac{1}{10}} = -9 + \frac{90}{9} = 1. \]

**Finite cardinality of power sets**

Given a set \( A \), we saw that the power set, \( P(A) \), always has larger cardinality than \( A \). If \( A \) is finite, we can say how many elements \( P(A) \) has.

As a basic fact, we can say that

\[ \| P(A) \| = 2^\| A \|. \]

In fact, mathematicians will extend this formula to levels of infinity, for example, \( \| \mathbb{R} \| = 2^\| \mathbb{N} \|. \)

At least as important as the facts, I think, are the connections with other facts and the underlying structure. So let’s take a look at this.

**Correspondence with Binary Numbers.** Take, for example, the set \( A = \{ 1, 2, 3, 4 \} \). The power set \( P(A) \) is the collection of all the subsets of \( A \), and we’d like to count how many subsets there are. We can describe each subset as a 4-term sequence of 0’s and 1’s. For example,

\[ 1, 0, 1, 1 \leftrightarrow \{ 1, 3, 4 \} \]

and

\[ 0, 1, 1, 0 \leftrightarrow \{ 2, 3 \}. \]

In particular, the first term of the sequence tells us whether the first element of the set is in the subset or not, and so on.

Once we have a 4-term sequence of 0’s and 1’s, we can pair these easily with all the 4-digit (4-bit, might be more appropriate) binary numbers. Since \( 1111_2 = 8 + 4 + 2 + 1 = 15 \), there must be 16 subsets of \( A \) (since we’re also counting \( 0000_2 \)). There are 32 5-digit binary numbers, 64 6-digit binary numbers, etc., so this agrees with the original fact.
Homework 13

1. We could define the number 0.9999… as the limit of the sequence
   \[ 0.9, 0.99, 0.999, 0.9999, \ldots \]
These numbers are clearly getting bigger and closer to 1, but do they get all the way to 1? This would be equivalent to the following sequence converging to 0:
   \[ 1 - 0.9, 1 - 0.99, 1 - 0.999, \ldots \]
   a. Simplify the terms of this last sequence.
   b. What is \( \lim_{i \to \infty} \frac{1}{10^i} \)?

2. Recall that for \( r < 1 \), a geometric series converges to
   \[ \sum_{i=0}^{\infty} ar^i = \frac{a}{1 - r} \]
   a. In Base 4, the expansion
      \[ 0.33333\ldots _4 = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots = \sum_{i=1}^{\infty} \frac{3}{4^i} \]
      Use the geometric series formula to find this sum. Don't forget that this series starts at \( i = 1 \), not \( i = 0 \).
   b. Do the same thing for \( 0.022222\ldots _3 \) in Base 3.

3. Write the following Base 3 numbers in Base 10.
   a. 12.223
   b. 0.0113
   c. 223

4. Write the following numbers in Base 3.
   a. \( \frac{3}{27} \)
   b. \( \frac{22}{27} \)
   c. \( \frac{42}{9} \)

5. Let \( A \) be the set of letters in the alphabet. How many elements are there in \( P(A) \)?

Answers: 1a) 0.1, 0.01, 0.001, … b) 0.
2a) 0.333… = \( -3 + 3 \cdot \frac{1}{1-\frac{1}{4}} = \frac{12}{9} = \frac{4}{3} \) 1. b) \( \frac{1}{3} \).
3a) 3 + 2 + \( \frac{2}{3} + \frac{2}{9} = 11 + \frac{2}{9} = 5.88888\ldots \)
   b) \( \frac{1}{9} + \frac{1}{27} = \frac{1}{9} = 0.148148148\ldots \)
   c) 6 + 2 = 8.
4a) 0.0023; b) \( \frac{22}{27} = \frac{18+3+1}{27} = 0.2113; \) c) \( \frac{42}{9} = \frac{27+9+6+0}{9} = 11.203 \).
5) \( P(A) = 2^{26} = 67,108,864 \).