MA 3260 Lecture 14 - Multiplication Principle, Combinations, and Permutations

Friday, November 2, 2018.

Objectives: Basic counting formulas

Here's another way of looking at the fact that a set with \( n \) elements has \( 2^n \) subsets.

If, for example, we have the set \( A = \{ 1, 2, 3 \} \), and we want to construct a subset \( S \). There are three things to consider:

1. Do we want \( 1 \in S \)?
2. Do we want \( 2 \in S \)?
3. Do we want \( 3 \in S \)?

The answer to each question is either Yes or No. In other words, each question has two choices. It may be natural for us to think that we should multiply the number of choices at each stage. For the first question, we choose Yes or No. For each of the first choices, there are two second choices. In other words, Yes-Yes and Yes-No, and also No-Yes and No-No. Two times two is four. For each of these, there are two third choices. Two times two times two is eight.

**Basic Principle 1.** *(Multiplication Principle)* If we have a two-stage process, where we have \( m \) choices for the first step, and for each of these choices, \( n \) choices for the second, then there are \( m \cdot n \) ways to do this.

**Example 1.** Suppose we have a bag of letter tiles, one tile for each letter in the alphabet. If we choose a first tile, then a second, how many ways can we do this?

If we don’t put the first tile back: For the first tile, we have 26 possibilities. For the second, we have 25 possibilities. This gives us \( 26 \cdot 25 = 650 \) combinations of first and second tiles.

If we put the first tile back (and so results like E-E are possible), then the number of possibilities is \( 26 \cdot 26 = 676 \).

**Quiz 14A**

How many car license plates are possible, if they have any three letters followed by any three numbers, like AFC 435?

**Permutations**

There are some basic counting formulas that we can use in more complicated situations. In mathematics, a permutation is a particular ordering of a collection of objects. The letters \( a, b, \) and \( c \), for example, can be ordered in the following ways:

\[
abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.
\]

In terms of the multiplication principle, we can say that there are 3 possibilities for the first thing, then 2 for the second, and 1 for the third. We’ll say

The number of permutations of three things is \( 3 \cdot 2 \cdot 1 = 6 \).

If there are four things to order, then we would expect there to be \( 4 \cdot 3 \cdot 2 \cdot 1 = 24 \) ways.

The number of permutations will always work out like this. If there are 10 things, the number of permutations will be \( 10 \cdot 9 \cdot 8 \cdots 3 \cdot 2 \cdot 1 = 3,628,800 \).
**Factorials.** A product of a number and all the positive integers smaller is called a *factorial*. We designate a factorial with a ! symbol. For example,

\[ 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120. \]

We’ll run into 0! from time to time, which doesn’t completely make sense. Generally, things work out best when multiplying nothing gives you 1. The standard definition makes

\[ 0! = 1. \]

**Permutations of \( r \) things taken from \( n \) things, \( nPr \)**

For example, consider the set \( \{a, b, c, d, e\} \). Suppose we take three elements from this set in sequence with no repeats (without putting them back). This is like the stuff we’ve been discussing. In this case, there are 5 possibilities for the first pick, then 4 possibilities for the second, and finally, 3 for the third. The total number ways that this can happen is

\[ 5P_3 = 5 \cdot 4 \cdot 3 = 60. \]

In general, we have the permutations formula

\[ nPr = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1). \]

This is basically \( n! \), but we leave off the last \( n-r \) factors. We can express this formula in terms of factorials as follows

\[ nPr = \frac{n!}{(n-r)!}. \]

For example, if we had a bag of tiles with the letters on the alphabet on them (each letter only once), and pulled out 7 tiles in sequence there would be

\[ 26P_7 = \frac{26!}{(26-7)!} = \frac{26!}{19!} = \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19!}{19!} = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 = 3,315,312,000 \text{ ways.} \]

Your calculator may have a permutations button, and for sure a factorial button. Unfortunately, factorials are too big for your calculator a lot of the time.

**Combinations of \( r \) things taken from \( n \) things, \( nCr \)**

In this context, a *combination* is an \( r \)-element subset taken from a set of \( n \) things. Each permutation of \( r \) things corresponds naturally to an \( r \)-element subset, but all the permutations of that subset are counted separately. In a combinations context, we only want to count the subset once. In particular, there are \( r! \) orderings of an \( r \)-element subset, so that’s how many times we’re counting each subset as permutations. Therefore,

\[ nCr = \frac{nPr}{r!} = \frac{n!}{r!(n-r)!}. \]

For example, if we were to take a handful of five alphabet tiles from the bag mentioned earlier, the number of possible handfuls would be

\[ 26C_5 = \frac{26!}{5!(26-5)!} = \frac{26!}{5!21!} = \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 21!} = \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 13 \cdot 5 \cdot 2 \cdot 23 \cdot 22 = 65,780 \]

**Quiz 14B**

1. Find \( _{100}C_97 \)
2. Find \( _8C_8 + _8C_7 + _8C_6 + \cdots + _8C_1 + _8C_0 \).
PASCAL’S TRIANGLE

The last quiz problem can be interpreted in another way. Consider the equation

\[ n C_r + n C_{r+1} = (n+1) C_{r+1}. \]

This is the relationship shown in the table below.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>15</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>21</td>
<td>35</td>
<td>35</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>28</td>
<td>56</td>
<td>70</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>36</td>
<td>84</td>
<td>126</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>45</td>
<td>120</td>
<td>210</td>
</tr>
</tbody>
</table>

Homework 14

1. Find the following
   a. 12!
   b. \( 9P_8 \)
   c. \( 9C_8 \)
   d. \( 100P_2 \)
   e. \( 25C_{18} \)

2. Find \( 9C_8 + 9C_7 + 9C_6 + \cdots + 9C_1 \).

3a. Find \( 7C_4 \), \( 7C_3 \), and \( 8C_4 \).
   b. Do you notice any relationship between these three numbers?
   c. Can you explain why this relationship should be true?

Answers: Quiz 14B: 1) 161,700. 2) These are all of the 8-element subsets, all the 7-element subsets, etc. In other words, these are all the subsets of an 8-element set. The answer is \( 2^8 = 256 \).

Homework 14: 1a) 479,001,600. b) 9. c) 9. d) 9900. e) 480700.

2) We're missing \( 9C_9 \) and \( 9C_0 \), which are both 1. Otherwise, we're talking about all of the subsets of a 9-element set, so the sum is \( 2^9 - 2 = 510 \).

3a) 35, 35, and 70. b) The pattern I'm wanting is \( 7C_4 + 7C_3 = 8C_4 \). c) Suppose you're talking about the subsets of \( \{1, 2, 3, 4, 5, 6, 7\} \) and \( \{1, 2, 3, 4, 5, 6, 7, 8\} \). Consider all of the 4-element subsets of the 8-element set. The ones that do not contain 8, are exactly the 4-element subsets of \( \{1, \ldots, 7\} \). The ones that do contain 8 are exactly the 3-element subsets of \( \{1, \ldots, 7\} \) with 8 thrown in.