Objectives: Introduce recurrence relations via the Fibonacci sequence.

Here’s a problem:

We have this robot soldier that is programmed to build a workstation for itself (this will take one month), and after the workstation is completed, to use the workstation to build a copy of itself (this also takes one month). Once the robot begins making copies of itself, it will continue to build one robot a month indefinitely. How big an army of robots will we have in one year?

Quiz 15A

Solve the problem.

This table might come in handy.

<table>
<thead>
<tr>
<th>time</th>
<th>0-month old robots</th>
<th>1-month old robots</th>
<th>older robots</th>
<th>total robots</th>
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<tr>
<td>$t = 0$</td>
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I would reason as follows: At $t = 0$, we have our first brand new robot, and that’s it. The first row would be $1\times0+0=1$.

At $t = 1$, our first robot will be one month old, and will have finished its workstation. The second row would be $0\times1+0=1$.

At $t = 2$, our first robot will be two months old, and will have finished building its first robot (a 0-month old robot). Therefore, the third row will be $1+0+1=2$. 


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**The Fibonacci Sequence**

Let’s review what we found. We started our “clock” at time $t = 0$ months. After a month we’re at $t = 1$ month, after two months, we’re at $t = 2$ months, and so forth. We’ll let $F_0$ be the number of robots at $t = 0$, $F_1$ the number of robots at $t = 1$, etc. In particular, $F_n$ will be the number of robots at time $t = n$ months.

At time $t = 0$, we’re setting our first robot into motion, and so we just have that one robot, and

$$F_0 = 1.$$  

At time $t = 1$, we still have our one robot, but it now has its workstation, and will begin to build another robot, but we only have that one complete robot, and

$$F_1 = 1.$$  

At time $t = 2$, our first robot has completed work on the second robot, and so we now have two robots, and

$$F_2 = 1 + 1 = 2.$$  

OK. At time $t = 3$, the first robot has built another robot, and the second robot has built its workstation. We now have three robots, and

$$F_3 = 2 + 1 = 3.$$  

At $t = 4$, things are starting to get going. The first two robots will have each built a new robot, and the third robot has completed its workstation, so

$$F_4 = 3 + 2 = 5.$$  

At $t = 5$, the two new robots from the previous month have completed their workstations, and the three older robots have each built a new robot, so now we have

$$F_5 = 5 + 3 = 8.$$  

You have this information in a chart, but just looking at the total number of robots, we have a nice pattern. The number of robots at the end of each month is equal to the sum of the totals from the previous two months. In other words, the robots around last month are still here, and the ones that were around two months ago will have each built a new robot. In mathematical symbols,

$$F_n = F_{n-1} + F_{n-2}.$$
We’ll come back to this, but we can now quickly answer the original problem.

\[(8)\quad F_6 = 5 + 8 = 13\]
\[(9)\quad F_7 = 8 + 13 = 21\]
\[(10)\quad F_8 = 13 + 21 = 34\]
\[(11)\quad F_9 = 21 + 34 = 55\]
\[(12)\quad F_{10} = 34 + 55 = 89\]
\[(13)\quad F_{11} = 55 + 89 = 144\]
\[(14)\quad F_{12} = 89 + 144 = 233\]

This sequence of numbers is called the Fibonacci Sequence. According to Wikipedia, Leonardo of Pisa (a.k.a. Fibonacci, or F-dog) wrote about this sequence, and so got his name attached to it, although it was written about much earlier in India. Finding \(F_{12}\) wasn’t too bad, but what if we wanted \(F_{200}\)? It would be nice, if we could just compute this directly. We can, but it’s an odd looking formula. This is called Binet’s Formula, named after Jacques Binet, although Wikipedia notes that Abraham DeMoivre knew about it earlier. I’ve also heard Euler knew about it too.

\[(15)\quad F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2}\right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{n+1}\right).\]

The interesting thing is that this last formula doesn’t look like it would be related to the Fibonacci sequence at all. How could that possibly always equal a whole number, for instance? The other big question would be, how could anyone figure out what the formula was? You couldn’t just guess it. Right?

**How would you find Binet’s Formula?**

Let’s back up a bit, and eventually, we’ll see that it’s not really that hard. This simpler example will give us some insight. Consider the sequence

\[(16)\quad 3, 6, 12, 24, 48, \ldots\]

Do you see the pattern? We start with 3, then multiply by 2, multiply by 2 again, etc. Each term is two times the term before. We can number the terms

\[(17)\quad A_0 \quad A_1 \quad A_2 \quad A_3 \quad A_4 \quad \ldots\]
\[\quad 3 \quad 6 \quad 12 \quad 24 \quad 48 \quad \ldots\]

The relation is that each term is two times the term before, which can be written

\[(18)\quad A_n = 2 \cdot A_{n-1}.\]

Is there an explicit formula for \(A_n\), one that doesn’t need to reference other terms in the sequence? Let’s look at the pattern. We know that

\[(19)\quad A_0 = 3.\]

Then we get each following terms by multiplying by 2.

\[(20)\quad A_1 = 3 \cdot 2\]
\[(21)\quad A_2 = 3 \cdot 2 \cdot 2\]
\[(22)\quad A_3 = 3 \cdot 2 \cdot 2 \cdot 2\]
\[(23)\quad A_4 = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2\]
Each term is a 3 times a bunch of 2's. In fact,

\begin{align*}
A_0 &= 3 \cdot 2^0 \\
A_1 &= 3 \cdot 2^1 \\
A_2 &= 3 \cdot 2^2 \\
A_3 &= 3 \cdot 2^3 \\
A_4 &= 3 \cdot 2^4
\end{align*}

In general, we must have

\begin{equation}
A_n = 3 \cdot 2^n.
\end{equation}

This is an example of an *explicit formula*. We can compute the value of any term without having to figure out the ones before it.

**Quiz 15B**

1. Find an explicit formula for the sequence

\begin{equation}
A_0 = 5, A_1 = 10, A_2 = 20, A_3 = 40, A_4 = 80, \ldots
\end{equation}

2. Find an explicit formula for this sequence

\begin{equation}
4, 12, 36, 108, 324, \ldots
\end{equation}

**The basic process**

OK. I want to look at finding an explicit formula for the Fibonacci sequence. We’ve seen some simple examples, like

\begin{equation}
3, 6, 12, 24, 48, \ldots
\end{equation}

where each term is 2 times the term before. In mathematical terminology, we might say that this sequence satisfied the *relational equation*

\begin{equation}
A_n = 2 \cdot A_{n-1}.
\end{equation}

Also important is that the first term is $A_0 = 3$. In a simple case like this, we can figure out that the explicit formula is

\begin{equation}
A_n = 3 \cdot 2^n.
\end{equation}

It turns out that if we have a simple relation like this, an explicit formula of the form

\begin{equation}
A_n = C x^n
\end{equation}

will describe any sequence that satisfies a relational equation like the one above.

**Example.** Consider the sequence

\begin{equation}
5, 15, 45, 135, \ldots,
\end{equation}

which satisfies the relations

\begin{equation}
A_n = 3 \cdot A_{n-1}, \text{ and } A_0 = 5.
\end{equation}

With a relation this simple, you can pretty much see the explicit formula, but I want to work through a process that will work in more complicated cases.

**Step 1.** Guess that the explicit formula will take the form

\begin{equation}
A_n = C x^n.
\end{equation}
Step 2. Plug this guess into the relational equation. Since $A_{n-1} = Cx^{n-1}$, we substitute for $A_n$ and $A_{n-1}$ to get
\begin{equation}
C x^n = 3 \cdot C x^{n-1}.
\end{equation}

Step 3. Solve for $x$. In this case, we can divide both sides by $C$ and $x^{n-1}$. This leaves us with
\begin{equation}
x = 3.
\end{equation}
Of course, we knew that already. Anyway, we now know that the explicit formula is
\begin{equation}
A_n = C \cdot 3^n
\end{equation}
for some number $C$.

Step 4. Solve for $C$, but using the value of $A_0$, which we know two ways. First, we’re given that
\begin{equation}
A_0 = 5,
\end{equation}
and second, we have the explicit formula
\begin{equation}
A_0 = C \cdot 3^0.
\end{equation}
Having $A_0$ two ways gives us the equation
\begin{align}
5 & = C \cdot 3^0 \\
5 & = C \cdot 1 \\
5 & = C.
\end{align}
This makes the explicit formula
\begin{equation}
A_n = 5 \cdot 3^n,
\end{equation}
which is what we expected.

Example. OK. Here’s an example similar to the ones we’ve done, but the explicit formula isn’t quite as obvious. The sequence is
\begin{align}
4, 6, 36, 54, 324, 486, 2916, 4374 \ldots
\end{align}
Specifically, the pattern is that each term is nine times the term that is two places before it. Do you see that? The relational equation is
\begin{equation}
A_n = 9 \cdot A_{n-2}.
\end{equation}
We want to find an explicit formula for this sequence. The trick is to guess that the explicit formula takes the form
\begin{equation}
A_n = C \cdot x^n.
\end{equation}
So just like before, we want to plug this guess into the relational equation. Note that $A_{n-2} = C \cdot x^{n-2}$. Therefore, we get
\begin{equation}
C \cdot x^n = 9 \cdot C \cdot x^{n-2}.
\end{equation}
Now, for $n = 10$, this last equation would be
\begin{equation}
C \cdot x^{10} = 9 \cdot C \cdot x^8,
\end{equation}
and if we divided both sides by $C$ and $x^8$, we would get
\begin{equation}
x^2 = 9.
\end{equation}
Similarly, for $n = 15$, we would have
\begin{equation}
C \cdot x^{15} = 9 \cdot C \cdot x^{13},
\end{equation}
and dividing by $C$ and $x^{13}$ would give us
\begin{equation}
x^2 = 9.
In general, if we divide \( C \cdot x^n = 9 \cdot C \cdot x^{n-2} \) by \( C \) and \( x^{n-2} \), we get

\[
(56) \quad x^2 = 9.
\]

This equation tells us that \( x \) must equal 3 or \(-3\). Going back to our guess for \( A_n \), we see that

\[
(57) \quad A_n = C \cdot (3)^n \quad \text{or} \quad A_n = C \cdot (-3)^n.
\]

are possible explicit formulas. It turns out that these aren’t the only possible explicit formulas, but we’re close. It may not be entirely obvious, but it turns out that the sum of any two explicit formulas also satisfies the same relational equation, and all the possible explicit formulas take the form

\[
(58) \quad A_n = B \cdot 3^n + C \cdot (-3)^n.
\]

All we have to do is to find the constants \( B \) and \( C \) that go with our sequence that starts with \( A_0 = 4 \) and \( A_1 = 6 \). We do this by setting up equations. We know \( A_0 \) two different ways. It’s equal to 4, and it’s also

\[
(59) \quad A_0 = B \cdot 3^0 + C \cdot (-3)^0 = B + C.
\]

Therefore, we know

\[
(60) \quad B + C = 4.
\]

We also know \( A_1 \) two different ways, \( A_1 = 6 \) and

\[
(61) \quad A_1 = B \cdot 3^1 + C \cdot (-3)^1 = 3B - 3C,
\]

and so we have another equation

\[
(62) \quad 3B - 3C = 6.
\]

The two equations together form a \textit{system of equations}

\[
(63) \quad B + C = 4 \quad \text{and} \quad (64) \quad 3B - 3C = 6.
\]

One way of solving a system of equations like this is to multiply the equations by numbers and adding the equations together. In this case, we could multiply the first equation by 3 to get

\[
(65) \quad 3B + 3C = 12 \quad \text{and} \quad (66) \quad 3B - 3C = 6.
\]

Adding the two equations together gives you

\[
(67) \quad 6B - 0C = 18,
\]

or

\[
(68) \quad B = 3.
\]

Plugging this into the equation \( B + C = 4 \), we see that

\[
(69) \quad C = 1.
\]

Therefore, our explicit formula must be

\[
(70) \quad A_n = 3 \cdot 3^n + 1 \cdot (-3)^n.
\]

As a quick check,

\[
(71) \quad A_2 = 3 \cdot 3^2 + 1 \cdot (-3)^2 = 3 \cdot 9 + 1 \cdot 9 = 27 + 9 = 36.
\]

**Quiz 15C**

1. Find \( A_3 \) using the explicit formula just derived.

2. Find \( A_4 \) and \( A_5 \).
OK. Now it’s your turn.

1. Consider the sequence
   \[ A_n = 4 \cdot A_{n-2} \]
   The first term is \( A_0 = 3 \), the second term is \( A_1 = 2 \), and the pattern is that each term is four times the term two before it. In other words,
   \[ A_n = 4 \cdot A_{n-2}. \]
   We start off guessing that the explicit formula will take the form
   \[ A_n = C \cdot x^n. \]
   a. We’ll need to know that \( A_{n-2} \). So \( A_{n-2} = C \cdot x^{n-2} \). Substitute into \( A_n = 4 \cdot A_{n-2} \), and simplify.
   b. Solve the equation in Problem 1 for \( x \).
   c. This gives us two possibilities for our guess of the explicit formula. What are these?
   d. It turns out that all possible explicit formulas will take the form
      \[ A_n = B \cdot 2^n + C \cdot (-2)^n. \]
      To solve for both \( B \) and \( C \), we’ll need to use \( A_0 = 3 \) and \( A_1 = 2 \). According to our possible explicit formulas (involving \( B \) and \( C \)), we have
      \[ A_0 = B \cdot 2^0 + C \cdot (-2)^0 = B + C \]
      \[ A_1 = B \cdot 2^1 + C \cdot (-2)^1 = 2B - 2C \]
      Since we need to have \( A_0 = 3 \) and \( A_1 = 2 \), we have two equations. The first equation is \( B + C = 3 \), and the second is \( 2B - 2C = 2 \). Dividing the second equation by 2, we get the system
      \[ B + C = 3 \]
      \[ B - C = 1 \]
      Solve it for \( B \) and \( C \).
   e. What is our explicit formula?

2. Consider the sequence
   \[ A_n = 5A_{n-1} - 6A_{n-2} \]
   which satisfies the pattern
   Find an explicit formula for this sequence.
1a) $x^2 = 4$.
   
b) $x = 2, -2$.
   
c) $A_n = C \cdot 2^n$ and $A_n = C \cdot (-2)^n$.
   
d) $B = 2$ and $C = 1$.
   
e) $A_n = 2 \cdot 2^n + 1 \cdot (-2)^n$.
   
2) Solve $x^2 - 5x + 6 = 0$ to get $x = 2, 3$. The general form of the explicit formula is $A_n = B \cdot 2^n + C \cdot 3^n$.

   Solve the system of equations
   \begin{align*}
   (82) & \quad B + C = 1 \\
   (83) & \quad 2B + 3C = 4
   \end{align*}

   You can multiply the first equation by either $-2$ or $-3$, and then add the equations. Or you can solve the first equation for either $B$ or $C$, and then substitute into the second equation. In any case, you should get $B = -1$ and $C = 2$.

   The explicit formula is $A_n = (-1) \cdot 2^n + 2 \cdot 3^n$. 