A quadratic equation
\[ ax^2 + bx + c = 0 \]
always has two solutions
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]
When \( b^2 - 4ac > 0 \), we get two real solutions. When \( b^2 - 4ac = 0 \), we get two real solutions that are equal (i.e., a double root). When \( b^2 - 4ac < 0 \), we get two complex solutions, and the square root is imaginary.

The two complex solutions always take the form \( r = P \pm Qi \), and so the two basic solutions of the recurrence relation are
\[ A_n = (P + Qi)^n \quad \text{and} \quad A_n = (P - Qi)^n. \]

The general solution would take the form
\[ A_n = B(P + Qi)^n + C(P - Qi)^n. \]
This is fine, except that the \( i \)'s complicate things a bit. It turns out that we can make everything real, but looking at it sideways.

It turns out that
\[ P + Qi = R \cos(\theta) + iR \sin(\theta), \]
where \( R = \sqrt{P^2 + Q^2} \), and \( \theta \) is the angle \( P + Qi \) makes with the positive real axis in the complex plane. The most important thing is something called DeMoivre’s Theorem, which says that when we raise a complex angle to a power \( n \), the distance from the origin increases by a power of \( n \), and the angle increases as a multiple of \( n \). In other words,
\[ (P + Qi)^n = R^n \cos(n\theta) + iR^n \sin(n\theta). \]

The other basic solution takes the similar form
\[ (P - Qi)^n = R^n \cos(n\theta) - iR^n \sin(n\theta). \]

The only problem is that these are complex functions. The last trick is to note that adding and subtracting these two basic solutions will give us two new basic solutions that work just as well. So we have
\[ (P + Qi)^n + (P - Qi)^n = 2R^n \cos(n\theta) \]
\[ (P + Qi)^n - (P - Qi)^n = 2iR^n \sin(n\theta). \]

Constant multiples don’t really matter, so we can just drop the 2 and \( 2i \). This gives us the alternative set of basic solutions
\[ A_n = R^n \cos(n\theta) \quad \text{and} \quad A_n = R^n \sin(n\theta). \]

The general solution in this case takes the form
\[ A_n = BR^n \cos(n\theta) + CR^n \sin(n\theta). \]

For example, consider the sequence
\[ 1, 2, 2, 0, -4, -8, -8, 0, 16, \ldots \]
described by the relational formula
\[ A_n = 2A_{n-1} - 2A_{n-2}. \]
Find an explicit formula for $A_n$.

The corresponding recurrence relation is

$$A_n - 2A_{n-1} + 2A_{n-2} = 0. \tag{14}$$

The characteristic equation is

$$x^2 - 2x + 2 = 0, \tag{15}$$

which has

$$x = -(-2) \pm \sqrt{4 - 4(1)(2)}2(1) = 1 \pm i. \tag{16}$$

If you plot this in the complex plane, you can see that the angle is $\theta = \frac{\pi}{4}$, and $R = \sqrt{2}$. Our two basic solutions are

$$A_n = (\sqrt{2})^n \cos\left(\frac{n\pi}{4}\right) \quad \text{and} \quad A_n = (\sqrt{2})^n \sin\left(\frac{n\pi}{4}\right). \tag{17}$$

The general solution is

$$A_n = B(\sqrt{2})^n \cos\left(\frac{n\pi}{4}\right) + C(\sqrt{2})^n \sin\left(\frac{n\pi}{4}\right). \tag{18}$$

To find $B$ and $C$, we use the first two terms of our sequence $A_0 = 1$ and $A_1 = 2$.

$$A_0 = B(\sqrt{2})^0 \cos\left(\frac{0\pi}{4}\right) + C(\sqrt{2})^0 \sin\left(\frac{0\pi}{4}\right) = B + 0 = 1 \tag{19}$$

$$A_1 = B(\sqrt{2})^1 \cos\left(\frac{1\pi}{4}\right) + C(\sqrt{2})^1 \sin\left(\frac{1\pi}{4}\right) = B + C = 2. \tag{20}$$

This gives us $B = 1$ and $C = 1$, and so our explicit formula is

$$A_n = (\sqrt{2})^n \cos\left(\frac{n\pi}{4}\right) + (\sqrt{2})^n \sin\left(\frac{n\pi}{4}\right). \tag{21}$$

As a quick check, consider

$$A_5 = (\sqrt{2})^5 \cos\left(\frac{5\pi}{4}\right) + (\sqrt{2})^5 \sin\left(\frac{5\pi}{4}\right) = 4\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) + 4\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = -8. \tag{22}$$

**Quiz 18A**

1. Consider the sequence

$$-1, 3, 1, -3, -1, 3, \ldots . \tag{23}$$

which satisfies the relational formula

$$A_n = -A_{n-2}. \tag{24}$$

Find the two basic solutions, the general solution, and the explicit formula for this sequence.

2. Find the two basic solutions for the following recurrence relations.

   a. $A_n - 2A_{n-1} - 8A_{n-2} = 0$.
   
   b. $A_n - 2A_{n-1} + 4A_{n-2} = 0$.
   
   c. $A_n + 6A_{n-1} + 9A_{n-2} = 0$.  

OK. So we have this thing called Pascal’s triangle, where every number in the chart is the sum of the two numbers just above it.

```
  1
 1 1
 1 2 1
 1 3 3 1
 1 4 6 4 1
 1 5 10 10 5 1
 1 6 15 20 15 6 1
 1 7 21 35 35 21 7 1
 1 8 28 56 70 56 28 8 1
 1 9 36 84 126 126 84 36 9 1
 1 10 45 120 210 252 210 120 45 10 1
```

Do these patterns show up anywhere else? Yes. Lot’s of places.

**Combinations.** How many subsets of size \( m \) are there in a set of size \( n \)?

To make the pattern complete, it’s convenient to consider empty sets as possibilities. How many empty sets are there? Just one. From there, let’s check that the number of subsets match up with Pascal’s triangle.

**Empty set.** How many subsets does the empty set have? Just one, the empty set.

**Set with one element.** Let’s take the set \( \{ a \} \). How many subsets does it have? There is one 0-element subset, \( \{ \} \), and one 1-element subset \( \{ a \} \).

**Set with two elements.** How many subsets does \( \{ a, b \} \) have? There’s the one 0-element subset, \( \{ \} \), two 1-element subsets, \( \{ a \}, \{ b \} \), and one 2-element subset, \( \{ a, b \} \). OK. These numbers agree with the top of Pascal’s triangle.

```
  \emptyset
  \emptyset a
  \emptyset a, b  ab
  \emptyset a, b, c  ab, ac, bc  abc
  \emptyset a, b, c, d  ab, ac, bc, ad, bd, cd  abc, abd, acd, bcd  abcd
```

We can see the pattern of Pascal’s triangle by noticing that each list includes all the sets directly above, plus the ones above and to the left with the new element added.

**Quiz 18B**

1. Look at Pascal’s triangle. What is \( 9C_3 \)?

2. What is \( 7C_4 \)?
Binomial coefficients

You may remember multiplying polynomials together in your high school math classes. For example, you might have done something like

\[(x + 1)^2 = (x + 1)(x + 1) = x^2 + x + x + 1 = x^2 + 2x + 1.\]

The word FOIL might come to mind. I mostly just want to see a pattern, so consider the product

\[(x + 1)^3 = (x + 1)(x + 1)(x + 1) = (x^2 + 2x + 1)(x + 1) = x^3 + x^2 + 2x^2 + 2x + x + 1 = x^3 + 3x^2 + 3x + 1.\]

Notice the coefficients in these two examples, 1-2-1 and 1-3-3-1. Do those look familiar? It turns out that if you expand out products of the form \((x + 1)^n\), the coefficients match rows in Pascal’s triangle. For example,

\[(x + 1)^7 = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1.\]

We’ve got all the powers of \(x\) starting with \(x^7\), and the coefficients come from Pascal’s triangle.

**Quiz 18C**

Expand the following using Pascal’s triangle.

3. \((x + 1)^4\).
4. \((x + 1)^5\).

**Homework 18**

1. Consider the sequence that satisfies the recurrence relation
   \[(29)\]
   \[A_n + A_{n-1} - 6A_{n-2} = 0,\]
   and starts with the terms \(A_0 = 1\) and \(A_1 = 1\).
   a. List out the first five terms of the sequence.
   b. What is the characteristic equation?
   c. Give the two basic solutions, and the general solution.
   d. Find the particular solution for this sequence.

2. Consider the sequence that satisfies the recurrence relation
   \[(30)\]
   \[A_n + 4A_{n-1} + 4A_{n-2} = 0,\]
   and starts with the terms \(A_0 = 1\) and \(A_1 = 1\).
   a. List out the first five terms of the sequence.
   b. What is the characteristic equation?
   c. Give the two basic solutions, and the general solution.
   d. Find the particular solution for this sequence.

3. Consider the sequence that satisfies the recurrence relation
   \[(31)\]
   \[A_n - \sqrt{3}A_{n-1} + A_{n-2} = 0,\]
   and starts with the terms \(A_0 = 1\) and \(A_1 = 1\).
   a. List out the first five terms of the sequence.
b. What is the characteristic equation?

c. Give the two basic solutions, and the general solution.

d. Find the particular solution for this sequence.

4. Consider the sequence that satisfies the recurrence relation

\[ A_n + 3A_{n-1} - 4A_{n-2} - 12A_{n-3} = 0, \]

and starts with the terms \( A_0 = 1, A_1 = 1, \) and \( A_2 = 1. \)

a. Multiply out \((x^2 - 4)(x + 3)\).

b. List out the first five terms.

c. What is the characteristic equation? (Note: You'll end up with a cubic equation.)

c. Give the three basic solutions, and the general solution.

d. Don't worry about the particular solution.

5. Reconstruct Pascal's triangle out to the row starting 1 10 ... 

6. Expand \((x + 1)^9\).

7. Expand \((x + a)^5\).

8. Expand \((x + 2)^5\).

9. Expand \((x - 2)^5\).

Answers on next page.
1) \(x^2 + x - 6 = 0\). Basic: \(A_n = 2^n\) and \(A_n = (-3)^n\). Particular: \(A_n = \frac{4}{3} \cdot 2^n + \frac{1}{3} \cdot (-3)^n\).

2) \(x^2 + 4x + 4 = 0\). Basic: \(A_n = (-2)^n\) and \(A_n = n(-2)^n\). Particular: \(A_n = (-2)^n - \frac{3}{2} n(-2)^n\).

3) \(x^2 - \sqrt{3}x + 1 = 0\). \(x = \frac{\sqrt{7}}{2} \pm \frac{i}{2}\). Polar: \(R = 1\) and \(\theta = \frac{\pi}{6}\). Basic: \(A_n = \cos\left(\frac{n\pi}{6}\right)\) and \(A_n = \sin\left(\frac{n\pi}{6}\right)\). Particular: \(A_n = \cos\left(\frac{n\pi}{6}\right) + (2 - \sqrt{3}) \sin\left(\frac{n\pi}{6}\right)\).

4) \(x^3 + 3x^2 - 4x - 12 = (x + 2)(x - 2)(x + 3) = 0\). Basic: \(A_n = (-2)^n\), \(A_n = 2^n\), and \(A_n = (-3)^n\). General: \(A_n = B(-2)^n + C2^n + D(-3)^n\).

6) \((x + 1)^9 = x^9 + 9x^8 + 36x^7 + 84x^6 + 126x^5 + 126x^4 + 84x^3 + 36x^2 + 9x + 1\).

7) \((x + a)^5 = x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5\).

8) \((x + 2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32\).

9) \((x - 2)^5 = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32\).