Note: Test III is Thursday (11/6/14).

1. Answer the following questions about cardinality.
   a. Who’s the mathematician that gave us the concept of cardinality? Georg Cantor
   b. If the cardinality of sets \(A\) and \(B\) is the same, what does that mean? That is, what is the definition? The elements of \(A\) and \(B\) can be paired off in a one-to-one correspondence.
   c. Illustrate how the elements in \(\mathbb{N}\) can be put into a one-to-one correspondence with the elements of \(\mathbb{Z}\).
      
      \[
      1 \leftrightarrow 0, \quad 2 \leftrightarrow -1, \quad 3 \leftrightarrow 1, \quad 4 \leftrightarrow -2, \quad 5 \leftrightarrow 2, \quad 6 \leftrightarrow -3, \quad 7 \leftrightarrow 3, \ldots
      \]
   d. What does it mean for a set to be countably infinite? That the set has the same cardinality as \(\mathbb{N}\)
   e. Do all uncountable sets have the same cardinality? No.

2a. The positive rationals, \(\mathbb{Q}^+\), is countable, which means that we can put the positive rationals into a sequence. List out the first 15 terms of a sequence that clearly hits all of \(\mathbb{Q}^+\). Repeats are OK. In addition, explain what the pattern is. One such sequence is \(
\left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots \right\}.
\) I started with the one fraction, whose numerator and denominator add up to 2. Then the two fractions, whose numerator and denominator add up to 3. Then the three that add up to 4, etc. All of the positive rationals will get into the list eventually.

2b. In our proof of the fact that the interval \((0, 1)\) is not countable, we assumed that elements of \((0, 1)\) could be put into a sequence, \(\{x_1, x_2, x_3, \ldots\}\). We then expressed each \(x_i\) in its decimal expansion. How did we construct a number from \((0, 1)\) that could not be in that sequence? We constructed a number \(y\), whose decimal expansion had the following property. The first digit is chosen to be different from the first digit of \(x_1\). Then the second digit of \(y\) was chosen to be different from the second digit of \(x_2\). In general, the \(i\)th digit of \(y\) is chosen to be different from the \(i\)th digit of \(x_i\). (Note: Technically, we wouldn’t want all 9’s, so we should avoid them.) This \(y\) can’t be equal to any \(x_i\), and so is not in the sequence.

3. Recall that \(\mu(A)\) represents the Lebesgue measure of \(A\).
   a. What is \(\mu\left((0, 2] \cup [5, 12] \cup \{52\}\right)\)? \(2 + 7 + 0 = 9\)
   b. What is \(\mu([0, 2] \cap [1, 10])\)? The intersection is the interval \([1, 2]\), so the measure is 1.
   c. What is \(\mu(\mathbb{Z})\)? \(0\)
   d. The rationals are countable, so they can be put into a sequence \(\{q_1, q_2, \ldots\}\). Let \(\epsilon > 0\) be any real number. What is \(\sum_{i=1}^{\infty} \mu((q_i - \frac{\epsilon}{2^i}, q_i + \frac{\epsilon}{2^i}))\). We have \(\epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{4} + \cdots = 2\epsilon\).
   e. Explain how Part d shows that \(\mu(\mathbb{Q}) = 0\). Clearly, \(\mathbb{Q}\) is a subset of the union of all those open intervals, and the union of the intervals has measure less than \(2\epsilon\) for \(\epsilon\) positive number \(\epsilon\). The only possible measure that is less than every positive number is zero.

4. The Cantor set can be defined as the set of all real numbers in \([0, 1]\) that has a Base 3 expansion using only 0’s and 2’s.
   a. Explain how the Cantor set must be uncountable. There is a natural one-to-one correspondence between the Base 3 expansions of the Cantor set and the Base 2 expansions of the reals in \([0, 1]\), just change the 2’s to 1’s. Since \([0, 1]\) is uncountable, the Cantor set must be also.
   b. What is the Lebesgue measure of the Cantor set? \(0\)
c. How would you write 0.020202₃ as a fraction (in Base 10)? This would be \( \frac{2}{3} + \frac{2}{27} + \frac{2}{243} = \frac{182}{243} \).


e. Write \( \frac{27}{81} \) as a Base 3 expansion. \( \frac{27+9+1}{81} = \frac{1}{3} + \frac{1}{9} + 181 = 0.1101₃ \)

5. Let \( A = \{1, 2, 3, \ldots, 20\} \).

a. How many elements does \( P(A) \) have? \( 2^{20} = 1,048,576 \).

b. Remember that the elements of \( P(A) \) are the subsets of \( A \). How many elements of \( P(A) \) contain 10? This is basically the same as the number of subsets that do not contain 10, so \( 2^{19} = 524,288 \).

c. How many elements of \( P(A) \) have \( \{4, 5, 6, \ldots, 20\} \) as a subset? This is the same as the number of subsets that contain only \( \{1, 2, 3\} \), so \( 2^3 = 8 \).

d. How many elements of \( P(A) \) have 19 elements? 20

e. How many elements of \( P(A) \) have four elements? \( 20C_4 = \frac{20!}{4!16!} = \frac{20 \cdot 19 \cdot 18 \cdot 17}{4 \cdot 3 \cdot 2 \cdot 1} = 4,845 \).

6a. You have 21 people in a club. How many ways can you choose a president, a vice president, and a treasurer (assume that these are different people)? \( 21 \cdot 20 \cdot 19 = 7,980 \).

b. In our 21-person club, how many ways can we choose a committee of three people? \( 21C_3 = \frac{21!}{3!18!} = 1,330 \).

c. How are \( 21C_6 \) and \( 21C_{15} \) related? They’re equal, \( \frac{21!}{6!15!} \).

d. What is \( 21C_0 + 21C_1 + 21C_2 + 21C_3 + 21C_4 + \ldots + 21C_{10} \) equal to? Since the number of 10-element subsets is the same as the number of 11-element subsets, and the number of 9-element subsets is the same as the 12-element subsets, etc., we’re talking about half of the subsets, or \( \frac{2^{21}}{2} = 1,048,576 \).

7. Consider the sequence

\[
\begin{align*}
1 &\quad 2 \quad 6 \quad 16 \quad 44 \quad 824 \quad 812 \quad 2596 \quad \ldots \\
A_n &= 5 \cdot A_{n-1} - 6 \cdot A_{n-2}.
\end{align*}
\]

a. Plug the guess \( A_n = C \cdot x^n \) into this last equation, and simplify. \( x^2 = 5x - 6 \) or \( x^2 - 5x + 6 = 0 \).

b. Solve for \( x \). (You should get two solutions.) \( x = 2, 3 \)

c. This gives us two possible explicit formulas for the sequence. These are what? \( A_n = C \cdot 2^n \) and \( A_n = C \cdot 3^n \).

d. I told you that all possible explicit formulas are linear combinations of the two you got in Part c. Therefore, the general form for an explicit solution is what? \( A_n = B \cdot 2^n + C \cdot 3^n \).

e. OK. So \( A_0 = -1 \) and \( A_1 = 2 \). Use these to solve for the coefficients in Part d. This gives us our explicit formula, which is what? \( A_n = -5 \cdot 2^n + 4 \cdot 3^n \).

8. If you have two objects \( A \) and \( B \) such that the combined shape \( A+B \) has the same shape and proportions as \( A \), the \( B \) is said to be an \textit{gnomon} for \( A \). We saw that the Golden Rectangle has a square gnomon, which is attached to the long side. Find the proportions of a rectangle that is its own gnomon. One way to do this is to attach the rectangles along their long sides. For simplicity, you may assume that the short side has length 1 and the long side has length \( x \). The short/long ratio for \( A \) is \( \frac{1}{x} \), and for \( A + A \), it is \( \frac{x}{2} \). Solving the equation \( \frac{1}{x} = \frac{x}{2} \) gives one positive solution \( x = \sqrt{2} \). The ratio, therefore, is \( \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \). In other words, a rectangle whose long side is \( \frac{1}{\sqrt{2}} \) times the length of its short side will be its own gnomon.