1. Generators and Relations

Last time, we saw that instead of defining the dihedral group $D_3$ as the symmetries of an equilateral triangle, we could have defined $D_3$ as the smallest group containing an element $s$ with order three, an element $t$ with order two, and satisfying the relation,

\[(1) \quad t \ast s = s^2 \ast t.\]

We could say that the relationship between $s$ and $t$ is such that an $s$ on the right of a $t$ is equivalent to an $s^2$ on the left of that $t$.

In this case, we’re going to say that $D_3$ is the group generated by $s$ and $t$ with relations

\[(2) \quad s^3 = e, \quad t^2 = e, \quad t \ast s = s^2 \ast t.\]

**Definition 1.** A group generated by a set of elements and a set of relations involving those elements will be the smallest and simplest group satisfying these conditions. By smallest, we mean that there are no other generators, and by simplest, we will only assume what must be true based on the relations. For example, if $s^4 = e$ is a relation, we would not assume that $s^2 = e$ is true.

1. Make a multiplication table for the group generated by $s$ and satisfying the relation $s^4 = e$.

2. Generators and Relations for the Dihedral Groups

The dihedral group $D_5$ would be the group of symmetries of a regular pentagon (all five sides equal and all five angles equal). Picture the pentagon with a vertex at the top and a side at the bottom. It has a rotation symmetry $\small{\text{â€”}}72^\circ$, which is a one-fifth rotation counter-clockwise. This should be a generator, just as $\small{\text{â€”}}120^\circ$ is a generator for $D_3$. If we let $s = \small{\text{â€”}}72^\circ$, then we must also have the relation

\[(3) \quad s^5 = e.\]

There are also five reflection symmetries, each with an axis of symmetry that passes through a vertex and the midpoint of the opposite side. One of these axes is completely vertical, so let’s call that one $\small{\text{|}}$, and we’ll let $t = \small{\text{|}}$. Clearly, $t$ is its own inverse, so we have the relation

\[(4) \quad t^2 = e.\]

Finally, we need a relation that tells how $s$ and $t$ interact. In parallel with what we’ve done before, what is $t \ast s$ equal to? Let’s think. This is a reflection followed by a one-fifth rotation counter-clockwise. Everything is backwards, because we’ve done a reflection, so this counter-clockwise rotation after the reflection, is like a clockwise rotation before. This suggests the relation

\[(5) \quad t \ast s = s^4 \ast t,\]

since $s^4 = s^{-1}$. In fact, the relation $t \ast s = s^{-1} \ast t$ would be fine. This relation tells us that we can simplify any product to the form $s^m \ast t^n$, so the only elements of $D_5$ must be

\[(6) \quad D_5 = \{ e, s, s^2, s^3, s^4, t, s \ast t, s^2 \ast t, s^3 \ast t, s^4 \ast t \}.\]
Here’s an example of a computation. We’ll move each of the three $s$’s past the $t$ on the left one at a time.

\[
t * s^3 * t = s^4 * t * s^2 * t \\
= s^4 * s^4 * t * s * t \\
= s^4 * s^4 * s^3 * t * t \\
= s^{12} * t^2 \\
= s^2
\]  

(7)

**Definition 2.** The dihedral group $D_n$ is the group generated by $s$ and $t$ with the relations

\[
s^n = e \\
t^2 = e \\
t * s = s^{n-1} * t
\]  

(8)

2. In $D_5$, simplify $(s^2 * t) * (s^3)$.

3. In $D_5$, simplify $t * s^2$.

4. How many elements does $D_5$ have? Is it 5!?

### 3. Homework 05

For problems 1-8, consider the group of symmetries for a square.

1. How many elements does this group have?

The symmetries of the equilateral triangle could be described with symbols \{1, 120, 240, \, \, \, \, \, |, \, \, \, \, \, \, \, /\}. We can use most of these symbols again (even though they might mean something a little different). In problems 2 and 3, list the symbols that can’t be used here in the most logical order.

2. Can’t use ____________,

3. and can’t use ____________

For problems 4-7, list the new symbols in the most logical order.

4. We’ll need ____________,

5. and we’ll need ____________

6. and we’ll need ____________

7. and finally, we’ll need ____________

8. Which of the symbols above (both new and old) corresponds to $s * t$, if $s = 90^\circ$ and $t = |$?

For problems 9-12, consider the group of symmetries of a 4in $\times$ 2in rectangle. The smallest rotation $s$ will be one generator, and one of the reflections is $t$.

9. What is the order of $s$?

10. What is the order of $t$?

11. Complete the relation $t * s =$ ____________

12. How many elements does this group have?
13. How many elements does the group of symmetries of the letter I have?

14. Does an isosceles triangle (that is not equilateral) have a rotation symmetry?

15. Does an isosceles triangle have a reflection symmetry?

16. How many elements are in the group of symmetries of an isosceles triangle (that is not equilateral)?

17. For the object shown in Figure 1, let the rotation generator be $s$. What is the order of $s$?

18. Does the object in Figure 1 have a reflection symmetry?

19. How many elements are in the group of symmetries for the object in Figure 1?

20. If we use the symmetry groups as a measure, which is more symmetric, a $2 \times 4$ rectangle or the object in Figure 1?

Bye.