1. ISOMORPHISMS PART II

In order to prove that a group $G$ is isomorphic to a group $H$, you must do the following.

1. Define a function $f : G \rightarrow H$.
2. Prove that $f$ is an isomorphism.
   (a) Prove that $f$ is one-to-one.
   (b) Prove that $f$ is onto.
   (c) Prove that $f(a \ast b) = f(a) \ast f(b)$ for all $a, b \in G$.

The function can also be $f : H \rightarrow G$, that is, from $H$ to $G$.

2. FUNCTIONS THAT ARE ONE-TO-ONE

Remember the definition of a function.

**Definition 1.** A function $f$ from a set $A$ to a set $B$, which we will indicate with $f : A \rightarrow B$, pairs every element of $A$ with a unique element in $B$.

Suppose $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4, 5\}$.

![Figure 1. A graphic representation of a function $f$.](image)

The picture in Figure 1, defines a function $f : A \rightarrow B$. Each arrow defines a function value. For example, $f(a) = 2$ and $f(d) = 3$. Note that the phrase “pairs every element of $A$ . . .” means that each element of $A$ must have an arrow coming from it. The phrase “with a unique element of $B$” means that there is no more than one arrow coming from any element of $A$. It would have been bad, for example, if we had that $f(a)$ was 1 or 2.

It’s perfectly fine, on the other hand, to have $f(c) = 3$ and $f(d) = 3$. Both $f(c)$ and $f(d)$ have single, well-defined values. The function value 3 does not have a unique pre-image, but that’s OK. In this case, I will say that the pre-image of 3 is $f^{-1}(3) = \{c, d\}$. Now the fact that $f(c) = 3$ and $f(d) = 3$ immediately eliminates the possibility that $f^{-1}$ is a function. For this and other reasons, we may wish to have functions that are one-to-one.

Now, the one thing we don’t want to see is two arrows pointing to the same function value. For an element $a$ in the domain, there is an arrow pointing from $a$ to $f(a)$. If two arrows point to the same function value, then there must be two elements $a$ and $b$ so that $f(a) = f(b)$. In a proof, the easiest thing to do is to consider two arrows pointing to the same function value, and then to show that they actually have to be the same arrow. That’s what we’ll use as our definition.

**Definition 2.** A function $f : A \rightarrow B$ is said to be one-to-one, if $f(a) = f(b)$ implies that $a = b$. 
The function \( f \) in Figure 1 is not one-to-one. The function \( g \) in Figure 2 is one-to-one.

![Image of functions](image)

**Figure 2.** A graphic representation of a function \( g \).

Again, when we wish to prove that a function is one-to-one, we will start with the assumption that \( f(a) = f(b) \) for elements \( a \) and \( b \) in the domain, and then show that \( a = b \) follows from this assumption.

To show that a function is not one-to-one, we will find one counterexample. For example, the function \( f \) above is not one-to-one, and even though \( f(c) = f(d) \), we see that \( c \neq d \).

A classic example of a function that is not one-to-one is the real-number function \( f : \mathbb{R} \rightarrow \mathbb{R} \) that satisfies \( f(x) = x^2 \). Since \( f(2) = f(-2) \), but \( 2 \neq -2 \), we have demonstrated that \( f \) is not one-to-one.

The function \( f : \mathbb{R} \rightarrow \mathbb{R} \) that satisfies \( f(x) = 2x - 3 \) is one-to-one. To prove this, we would consider two generic elements \( a \) and \( b \) in the domain and assume that \( f(a) = f(b) \). This would mean that \( 2a - 3 = 2b - 3 \). By the uniqueness of the binary operation \(+\) over the reals, if we add 3 to both sides, the two sides must still be equal. In particular \( 2a = 2b \). Similarly, multiplying both sides by \( \frac{1}{2} \) implies \( a = b \). In other words, we have just shown that assuming \( f(a) = f(b) \) leads us to the conclusion \( a = b \), and \( f \) is one-to-one.

3. Functions that are Onto

In the function \( f \) defined by Figure 1, the elements \( 4, 5 \in B \) are not paired with anything in \( A \). The function \( g \) in Figure 2 doesn’t have anything paired with \( 4 \in B \). We will say that neither of these functions is onto.

**Definition 3.** A function \( f : A \rightarrow B \) is onto, if every element of \( B \) is paired with at least one element in \( A \).

In other words, every element of \( B \) must have an arrow pointing to it. Since arrows point from \( a \) to \( f(a) \), if we have an element \( b \in B \), we want \( f(a) = b \) for some \( a \). Our logical definition will use this.

**Definition 4.** A function \( f : A \rightarrow B \) is onto, if \( b \in B \) then it follows that there must be an \( a \in A \) such that \( f(a) = b \).

To prove that a function is not onto, we will exhibit one counterexample. For example, the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = x^2 \) is not onto, \(-4 \in \mathbb{R} \) (the range), but there is no \( a \) such that \( f(a) = -4 \), since \( a^2 \geq 0 \) for all \( a \) in the domain.

To prove that a function is onto, we’ll consider a generic element in the range, and find something that maps to it. The function \( f(x) = 2x - 3 \) is onto. Consider the element \( b \in \mathbb{R} \). We’ll do some detective work first. We need \( f(a) = b \). This is the same as \( 2a - 3 = b \). In order for this to happen, we need \( 2a = b + 3 \) and \( a = \frac{b+3}{2} \). OK. We now now what \( a \) needs to be. We also know that for any \( b \in \mathbb{R} \), \( b + 3 \in \mathbb{R} \), since we can always add 3 to a number, and also \( \frac{b+3}{2} \in \mathbb{R} \), since we can always divide by 2. To check, we see that

\[
(1) \quad f \left( \frac{b+3}{2} \right) = 2 \left( \frac{b+3}{2} \right) - 3 = (b+3) - 3 = b.
\]

This proves that for any \( b \), there is an element in the domain, specifically \( a = \frac{b+3}{2} \), that maps to it.

Finally, we can be explicit in defining a one-to-one correspondence.
Definition 5. A function $f : A \rightarrow B$ is a **one-to-one correspondence**, if $f$ is one-to-one and onto.

4. Some examples

Consider the following functions $f : \mathbb{R} \rightarrow \mathbb{R}$ (maybe not defined on all of $\mathbb{R}$). Determine which ones are 1-1 and/or onto.

1. $f(x) = e^x$.
2. $f(x) = x^3$.
3. $f(x) = \sin(x)$.
4. $f(x) = \frac{1}{x}$.
5. $f(x) = \tan(x)$.

5. Homework 07

1. Consider all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(5) = f(10)$.
   (a) None of these functions are one-to-one    (b) Some of these functions are one-to-one, but not all
   (c) All of these functions are one-to-one    (d) No such function exists    (e) none of these

2. Consider all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(5) = f(5)$.
   (a) None of these functions are one-to-one    (b) Some of these functions are one-to-one, but not all
   (c) All of these functions are one-to-one    (d) No such function exists    (e) none of these

3. Consider all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = f(x)$ for all $x \in \mathbb{R}$.
   (a) None of these functions are one-to-one    (b) Some of these functions are one-to-one, but not all
   (c) All of these functions are one-to-one    (d) No such function exists    (e) none of these

4. Consider all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that whenever $f(x) = f(y)$, then $x = y$.
   (a) None of these functions are one-to-one    (b) Some of these functions are one-to-one, but not all
   (c) All of these functions are one-to-one    (d) No such function exists    (e) none of these

5. Consider all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that whenever $y$ is an element of $R$, there is an element $x$ such that $f(x) = y$.
   (a) None of these functions are onto    (b) Some of these functions are onto, but not all
   (c) All of these functions are onto    (d) No such function exists    (e) none of these

6. Consider all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = 5$ for some $x$ in $\mathbb{R}$.
   (a) None of these functions are onto    (b) Some of these functions are onto, but not all
   (c) All of these functions are onto    (d) No such function exists    (e) none of these

7. Consider all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) \neq 5$ for every $x \in \mathbb{R}$.
   (a) None of these functions are onto    (b) Some of these functions are onto, but not all
   (c) All of these functions are onto    (d) No such function exists    (e) none of these
8. Consider all onto functions \( f : D_3 \to D_3 \).

(a) None of these functions are one-to-one  
(b) Some of these functions are one-to-one, but not all  
(c) All of these functions are one-to-one  
(d) No such function exists  
(e) none of these

Georg\(^1\) Cantor defined two sets to be of the same *cardinality*, if there was a one-to-one correspondence between them. This was his way of determining whether two sets were the same size.

Consider the function \( f : \mathbb{N} \to \mathbb{Z} \), (where \( \mathbb{N} = \{1, 2, 3, 4, \ldots\} \)) defined by

\[
(2) \quad f(n) = \frac{(1 - 2i)(-1)^n - 1}{4}.
\]

This function is one-to-one. I can prove it. I’ll spare you the details.

9. Is this function onto? Hint: plug a bunch of natural numbers into this function to see what’s going on.

(a) yes  
(b) no, it misses 0  
(c) no, it misses 5  
(d) it doesn’t miss anything, but it’s just not onto  
(e) none of these

10. According to Cantor, are \( \mathbb{N} \) and \( \mathbb{Z} \) the same size?

(a) yes  
(b) no, \( \mathbb{Z} \) is definitely bigger  
(c) no, \( \mathbb{N} \) is definitely bigger  
(d) no, neither one is bigger, but they’re still not the same size  
(e) none of these

Bye.

\(^1\)I used to pronounce this “George,” but after a bobsled-watching epiphany, I now call the guy “GAY org”