1. The Conway Polynomial

We’ve seen how we can transfer the information contained in a knot into the knot group. Knot groups are easier to work with than knots, but essentially all of the information available transfers over, so there is too much information in the knot group to handle very easily. Even the trefoil knot group is very complicated, and it’s hard to work with. We need to lose some information. That sounds bad, but it’s probably the most important thing we do in mathematics. There is great power in losing information skillfully.

In 1923, J.W. Alexander found a way of extracting useful information from the knot group and depositing it in a polynomial. This is an incredibly wild idea that works really well. Knot theorists are still finding new and different ways of attaching polynomials to knots, and this is probably the most fruitful method for solving problems concerning knots. In 1970, John Conway found a nice way of computing knot polynomials that bypasses the knot group. Conway is one of the great mathematicians of the 20th century. I didn’t see the movie, but I read *A Beautiful Mind* about John Forbes Nash. At Princeton in the 1930’s, the students and faculty used to hang out in the department lounge and play games. These games led to a lot of significant ideas. Nash and Conway were participants, and I believe people like Shapley, Shubik, Banzhaf, and Arrow were in this crowd too.

This is how you find Conway’s polynomials. First, in addition to knots, we’re going to throw in things called links. A link is simply a collection of knots considered together as one object. Figure 1 shows two examples of links. One is a pair of unknots, and the other is an unknot together with a trefoil. Links don’t actually have to link.

![Figure 1. These are two examples of links.](image)

In addition, we are going to place an orientation on each knot and link, so the Conway polynomials will actually be assigned to oriented knots and links.

OK. Here are the rules.

**Rule 1.** To each oriented knot or link $K$, there will be associated a polynomial with integer coefficients $\nabla_K$.

**Rule 2.** If $K$ is the unknot, then $\nabla_K = 1$.

**Rule 3.** If diagrams for three knots differ only at one crossing, where at this crossing, knot $R$ looks like the *right-handed* crossing shown in Figure 2, knot $L$ looks like the *left-handed* crossing shown in Figure 2, and knot $S$ looks like the *smoothed* crossing shown in Figure 2, then the three corresponding Conway polynomials are related by

\[
\nabla_R - \nabla_L = x \cdot \nabla_S.
\]

This may sound a bit odd, but it is really quite brilliant. An example or two should make this clear.

Consider the knots shown in Figure 3. Let’s say we started with an unknot and the diagram on the left of Figure 3. The arrows indicate an orientation on this knot, and if you compare this crossing with Figure 2, you’ll see that this is a right-handed crossing.
We can change this crossing in two ways. One way is to drop the over-crossing to an under-crossing. This gives us the diagram in the middle of Figure 3, and now the crossing is left-handed. The other way is to remove the crossing altogether. This gives us the diagram on the right. Note that both of these changes leaves the rest of the knot unchanged (including the orientation). The three knots we’ll label as $R$, $L$, and $S$.

Rule 3 tells us that the Conway polynomials for these knots are related by the equation

\[ \nabla_R - \nabla_L = x \cdot \nabla_S. \]

Both $R$ and $L$ are unknots, so Rule 2 tells us that their Conway polynomials must be $\nabla_R = 1$ and $\nabla_L = 1$. The equation is now

\[ 1 - 1 = x \cdot \nabla_S. \]

It follows that

\[ \nabla_S = 0. \]

In other words, the Conway polynomial for a pair of unlinked unknots is 0. It turns out that unlinked links of any kind will have Conway polynomial equal to 0.

In Figure 4, we have three more knot diagrams differing at one crossing. This crossing is right-handed on the first, left-handed on the middle one, and smoothed on the right. Here $R$ is an unlinked link, $L$ is a linked link, and $S$ is an unknot. We know two of these, so the equation from Rule 3 tells us

\[ 0 - \nabla_L = x \cdot 1. \]
It follows that $\nabla L = -x$. This particular knot is called a *simple link*. There are actually two, a left-handed one, and a right-handed one. This one is left-handed.

![Knot Diagrams](image)

**Figure 5.** Three oriented knot diagrams differing at one crossing.

2. **Homework 12**

OK. I’m going to have you find the Conway polynomial for one of the trefoils. Refer to Figure 5

1. In Figure 5, what knot is the knot $R$?
   (a) The unknot.  (b) A trefoil.  (c) A simple link.  (d) A figure-eight knot.  (e) none of these

2. Therefore, what is $\nabla_R$?
   (a) $\nabla_R = 1$  (b) $\nabla_R = x$  (c) $\nabla_R = -x$  (d) $\nabla_R = x + 1$  (e) none of these

3. What knot is knot $L$?
   (a) A trefoil.  (b) The unknot.  (c) A simple link.  (d) A figure-eight knot.  (e) none of these

4. What link is link $S$?
   (a) A simple link like the one in Figure 4.  (b) A simple link different from the one in Figure 4.  (c) A doubly linked link.  (d) An unlinked link.  (e) none of these

5. Therefore, what is $\nabla_S$?
   (a) $\nabla_S = -x$  (b) $\nabla_S = 1$  (c) $\nabla_S = x$  (d) $\nabla_S = x^2 + 1$  (e) none of these

6. What is $\nabla_L$?
   (a) $\nabla_L = x^2 + 1$  (b) $\nabla_L = x$  (c) $\nabla_L = -x$  (d) $\nabla_L = x^2 - 1$  (e) none of these
The Conway polynomial is an example of a knot invariant. Actually it’s an oriented knot invariant. What this means is that if two diagrams represent the same knot with the same orientation, then they will give you the same Conway polynomial.

7. Does the fact that the Conway polynomial is an oriented knot invariant mean that if two oriented knots have the same Conway polynomial, then they must actually be the same knot?

8. If two knot diagrams give you two different Conway polynomials, can the two knots be the same knot?

9. Based only on what I’ve told you (in the last several sentences), can the Conway polynomial be used to prove that two knot diagrams represent two different knots?

10. Can the Conway polynomial be used to prove that two knot diagrams represent the same knot?