1. The Fundamental Homomorphism Theorem

Given a group $G$, it turns out that the subgroups of $G$ and the homomorphisms on $G$ almost contain the same information. Let me review cosets and kernels a bit, and then I’ll explain what I mean.

What we know about the cosets of $H$, a subgroup of $G$.

- $H = eH$ is one of the left cosets.
- Any two cosets of $H$ are the same size as $H$.
- Given two cosets $aH$ and $bH$, either $aH = bH$ (as sets) or $aH \cap bH = \emptyset$.

What we know about the kernel of a homomorphism $f : G \to J$.

- $\ker(f)$ is a subgroup of $G$.
- If $f(a) = x$, then $aK = f^{-1}(x)$, where $K = \ker(f)$. That is, the left cosets of the kernel are precisely the preimages of elements in $J$.

Let’s look at the last item as a review.

Let $f : G \to J$ be a homomorphism, $K$ the kernel of $f$, and $a$ an element of $G$. Certainly, $f(a)$ is some element of $J$, and let’s call it $x$. I want to show that $aK = f^{-1}(x)$. Note that both $aK$ and $f^{-1}(x)$ are subsets of $G$. We want to show that they contain precisely the same elements.

Show $aK \subset f^{-1}(x)$. Let $ak \in aK$. Then

\[ f(ak) = f(a)f(k) = xe = x. \]

This means that $ak \in f^{-1}(x)$. Since $ak$ is a generic element of $aK$, all the elements in $aK$ also belong to $f^{-1}(x)$.

Show $f^{-1}(x) \subset aK$. Let $b \in f^{-1}(x)$. Then $f(b) = x$. We need to find a $k \in K$ so that $b = ak$. Since $f(a) = x$ also, we know that

\[ f(a^{-1}b) = f(a)^{-1}f(b) = x^{-1}x = e, \]

so $a^{-1}b \in K$. Let’s call this element $k$, so

\[ a^{-1}b = k, \]

and

\[ b = ak \in aK. \]

**Basic Principle 1.** When we have a homomorphism we have a nice correspondence between cosets of the kernel and elements in the image. In particular,

\[ aK \longleftrightarrow f(a). \]

2. The Correspondence

OK. Let’s take this a bit further. If we have two elements $a, b \in G$, then the correspondence is

\[ abK \longleftrightarrow f(ab). \]

Since $f(ab) = f(a)f(b)$, this suggests an operation $aK \cdot bK = abK$.

Now, we can define an operation pretty much any way we want, and this particular operation will look just like the operation in $G$. The one thing we have to check is to make sure that this really is a binary operation. That is, we want to make sure that when we multiply two cosets, we only get one answer. For example, if
\[ aK = a'K \text{ and } bK = b'K, \text{ then } aK \cdot bK = abK \text{ and } a'K \cdot b'K = a'b'K. \text{ If } abK \neq a'b'K, \text{ then we've gotten two different answers from the same two cosets.} \]

Here's a preliminary lemma.

3. QUIZ

Suppose \( K \) is the kernel of a homomorphism \( f : G \to J \). Let \( k \in K \) and \( a \in G \). Show that there is an element \( k' \in K \) such that \( ka = ak' \).

1. To satisfy \( ka = ak' \), what must \( k' \) be?
2. Show that your \( k' \) is an element of \( K \).

4. WELL-DEFINEDNESS

The property that an operation (or functions in general) are single valued is usually referred to as the operation is well-defined. We want to prove the following.

**Theorem 1.** Let \( f : G \to J \) be a homomorphism, and let \( K \) be the kernel of \( f \). If \( aK = a'K \) and \( bK = b'K \), then \( abK = a'b'K \).

**Proof.** If we can show that \( abK \subset a'b'K \), then showing \( a'b'K \subset abK \) would be exactly the same. Let \( abk_1 \) be an element of \( abK \) (i.e., let \( k_1 \in K \)). Since \( aK = a'K \) and \( bK = b'K \), there must be elements \( k_2, k_3 \in K \) such that \( a = a'k_2 \) and \( b = b'k_3 \). Therefore,

\[ abk_1 = a'k_2b'k_3k_1. \]

By the quiz problem, we know that there is some element \( k_2' \in K \) so that \( k_2b' = b'k_2' \). Therefore,

\[ abk_1 = a'k_2b'k_3k_1 = a'b'k_2'k_3k_1 \in a'b'K, \]

since \( k_2'k_3k_1 \in K \). We’re done. \( \square \)

5. HOMEWORK 21

Let \( f : G \to J \) be a homomorphism, and let \( K \) be the kernel of \( f \).

1. Suppose \( x \in aK \). Then there is an element \( k \in K \) such that
   (a) \( x = k \quad (b) \ a = k \quad (c) \ k \notin K \quad (d) \ x = ak \quad (e) \) none of these

2. Let \( k' = aka^{-1} \) (same \( k \) as the one in problem 1). Then \( f(k') = \ldots \)
   (a) \( aea^{-1} \quad (b) \ aa^{-1}k \quad (c) \ f(a)f(k)f(a^{-1}) \quad (d) \ aka^{-1} \quad (e) \) none of these

3. Furthermore, \( f(k') = \ldots \)
   (a) \( ak \quad (b) \ ka \quad (c) \ ea \quad (d) \ f(a)e(f(a))^{-1} \quad (e) \) none of these

4. Therefore, \( k' \) is \ldots
   (a) in \( f(K) \quad (b) \text{ in } K \quad (c) \text{ in } aK \quad (d) \text{ in } Ka \quad (e) \) none of these

5. Note that \( k'a = \ldots \)
   (a) \( ka \quad (b) \ k \quad (c) \ a \quad (d) \ x \quad (e) \) none of these
6. We started with the assumption that \( x \in aK \). Now we know it’s also in . . .
(a) \( a^{-1}K \)   (b) \( Ka^{-1} \)   (c) \( Ka \)   (d) \( K \)   (e) none of these

7. We now know that \( aK \subset Ka \). We could just as easily show that \( Ka \subset aK \). Therefore,
(a) \( aK = K \)   (b) \( Ka = K \)   (c) \( K = G \)   (d) \( aK = Ka \)   (e) none of these

Definition 1. A subgroup \( H \) of a group \( G \) is said to be normal, if \( aH = Ha \) for every \( a \in G \).

8. What do problems 1-7 show?
(a) The kernel of a homomorphism is always a normal subgroup.
(b) Any subgroup called \( K \) is normal.
(c) Every subgroup of \( G \) is normal.
(d) The problems don’t show anything, but I showed my doggie at Westminster.
(e) none of these

As with cosets, we can define \( aHb = \{ ahb \mid h \in H \} \).

9. Suppose \( H \) is a normal subgroup. If \( h \in H \), then \( aha^{-1} . . . \)
(a) is equal to some \( h' \in H \).   (b) is equal to \( h \).   (c) is equal to \( e \).   (d) \( a^{-1}ha \)   (e) none of these

10. Suppose \( H \) is a subgroup of \( G \), and \( aHa^{-1} = H \) for every \( a \in G \). Then \( H \) is . . .
(a) a normal subgroup of \( G \).   (b) a kernel of \( G \).   (c) equal to \( G \).   (d) a homomorphism.
(e) none of these