1. In the figure below is a trefoil $T$ and a figure-eight knot $F$. Draw the connect sum of these two knots $T \# F$.

This would take some work for me to draw, and I’m guessing you all can do this one.

2. We know that the Conway polynomial of $T$ is $\nabla_T = 1 + x^2$ and $\nabla_F = 1 - x^2$. What is the Conway polynomial for $T \# F$? Show your work and explain what you’re doing.

$$\nabla_{T \# F} = \nabla_T \cdot \nabla_F = (1 + x^2)(1 - x^2) = 1 - x^4$$

3. We do not know that the unknot is the only knot with a Conway polynomial $\nabla_K = 1$. There may be others. Suppose $A \# B = U$, where $U$ is the unknot. Do you know what $\nabla_A$ is? Explain how you know.

Conway polynomials are polynomials with integer coefficients. Since the degrees add, when polynomials are multiplied, and 1 has degree zero, $\nabla A$ and $\nabla B$ must have degree zero. Degree zero polynomials are constants, so $\nabla A$ and $\nabla B$ must be integers. They also must be multiplicative inverses, and so $\nabla A = \pm 1$. 
4. Shown below, the Mystery Knots A and B taken together form an unlinked link C. Show using the Conway polynomial equation that $\nabla_C = 0$.

$\nabla_R - \nabla_L = x\nabla_S$

The right- and left-handed versions of this link are the same (flip B over and then over again). Therefore, you get $\nabla_R - \nabla_L = 0 = x \cdot \nabla_C$, and $\nabla_C = 0$.

5. Draw a picture of a trefoil. Draw two non-trivial loops a and b (your choice). Now draw $a^2b^{-1}$. Loops are oriented, so don’t forget your arrows!

I would draw a through one leaf of the trefoil, and b through another. The loop $a^2b^{-1}$ would go through the one leaf twice, and then through the second leaf backwards once.
6a. Consider a six-hour clock (we’re talking \( \mathbb{Z}_6 \), and 6 is at the top). Let \( j \) represent jumps of \( j \) hours. For each \( j \), let \( k \) be the number of jumps it takes to start at 6 o’clock and get back to 6 o’clock. For each \( j = 1, 2, 3, 4, 5, 6 \), find the corresponding \( k \).

For \( j = 1 \), \( k = 6 \). For \( j = 2 \), \( k = 3 \). For \( j = 3 \), \( k = 2 \). For \( j = 4 \), \( k = 3 \). For \( j = 5 \), \( k = 6 \). For \( j = 6 \), \( k = 1 \).

6b. Explicitly describe a process that will find \( k \) for any positive integer. You may use language such as “Divide by eight, take the remainder, subtract two, . . . ” You may also refer to your answer in problem 6a.

Were assuming that were in \( \mathbb{Z}_6 \). Given \( j \), find the integer 1, 2, 3, 4, 5, or 6, that is equivalent modulo 6 to 6. That is, divide by 6, and take the smallest positive remainder. Call this \( j \). Find the greatest common divisor for \( j \) and 6. Call this \( d \). Then \( k = 6/d \).

7. List out all the subgroups of \( \mathbb{Z}_{15} \). For each subgroup, list out all the left cosets. Express your answers in set notation (e.g. \{ 1, 5, 9, 11 \}).

\[
\begin{align*}
\{ 0 \} & : \{ 0 \}, \{ 1 \}, \{ 2 \}, \{ 3 \}, \{ 4 \}, \{ 5 \}, \{ 6 \}, \{ 7 \}, \{ 8 \}, \{ 9 \}, \{ 10 \}, \{ 11 \}, \{ 12 \}, \{ 13 \}, \{ 14 \}.
\{ 0, 5, 10 \} & : \{ 0, 5, 10 \}, \{ 1, 6, 11 \}, \{ 2, 7, 12 \}, \{ 3, 8, 13 \}, \{ 4, 9, 14 \}.
\{ 0, 3, 6, 9, 12 \} & : \{ 0, 3, 6, 9, 12 \}, \{ 1, 4, 7, 10, 13 \}, \{ 2, 5, 8, 11, 14 \}.
\end{align*}
\]

\( \mathbb{Z}_{15} : \mathbb{Z}_{15} \).
8. Let \( G \) be a group, and suppose \( H \) is a subset. Consider the following theorem. If \( H \) is non-empty and for any pair of elements \( a, b \in H \) we know that \( ab^{-1} \in H \), then \( H \) is a subgroup. Fill in the blanks in this proof.

There is some element \( a \in H \), because \( H \) is non-empty.

Therefore, \( a, a^{-1}, e \in H \), so \( a^{-1} = \) must be in \( H \).

Similarly, \( e, a \in H \), so \( ea^{-1} = a^{-1} \) must be in \( H \).

It follows that for any \( a \in H \), \( a^{-1} \in H \).

Suppose \( a, b \in H \). We know that \( a^{-1} \in H \).

Therefore, \( a, b^{-1}, ab^{-1} \in H \), and so \( ab^{-1} = ab \in H \).

We have shown that the operation is closed in \( H \), the identity is in \( H \), and for each element of \( H \), its inverse is also in \( H \).

Therefore, \( H \) must be a subgroup of \( G \).

9. Draw the subgroup lattice for \( D_4 \).

The thing in the lecture notes was wrong. It should look like