Math 3361-Modern Algebra — Practice Test II with answers in red

10/08/07

1. Consider the set \( \mathbb{Q}^+ = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}^+ \text{ and } m \text{ and } n \text{ are relatively prime} \right\} \). In both cases, the superscript + indicates a restriction to the positive rationals and the positive integers. Here the rational numbers are the collection of all fractions reduced to lowest terms (including fractions like \( \frac{2}{1} \), which we’ll just call 7 for convenience). Multiplication will be the usual multiplication. You may assume that this operation is associative.

a. Is there an identity? If so, what is it? Yes. \( \frac{1}{1} = 1 \).

b. Suppose \( \frac{m}{n} \in \mathbb{Q}^+ \). What is the inverse of this element? \( \frac{n}{m} \).

c. Is \( \left\langle \mathbb{Q}^+, \cdot \right\rangle \) a group? Yes. The operation is good. It’s associative, it has an identity, and every element has an inverse. We know all of this stuff from high school algebra.

d. Find the prime factorization for \( \frac{12}{25} \). \( \frac{2^2 \cdot 3}{5^2} = 2^2 3^1 5^{-2} \)

Let \( f : \mathbb{Q}^+ \rightarrow \mathbb{Q}^+ \) be defined such that \( 2^a 3^b 5^c \mapsto 2^a 3^b 5^c \). Specifically, every factor of 2 maps to a factor of 3, and every factor of 3 maps to a factor of 2. Every other factor is left alone. For example, \( f(20) = f(2^2 \cdot 5) = 3^2 \cdot 5 = 45 \).

e. What is \( f(6) \)? \( f(2^1 \cdot 3^1) = 2^1 \cdot 3^1 = 6 \)

f. What is \( f(45) \)? \( f(2^0 \cdot 3^2 \cdot 5^1) = 2^2 \cdot 3^0 \cdot 5^1 = 20 \)

g. What is \( f(24/5) \)? \( f(2^3 \cdot 3^1 \cdot 5^{-1}) = 2^1 \cdot 3^1 \cdot 5^{-1} = \frac{54}{5} \)

h. To show that \( f \) is one-to-one, we would start with the assumption that \( f(q_1) = f(q_2) \). Suppose \( q_1 = 2^{a_1} 3^{a_2} 5^{c_1} \cdot \ldots \) and \( q_2 = 2^{a_2} 3^{b_2} 5^{c_2} \cdot \ldots \). What are \( f(q_1) \) and \( f(q_2) \)? \( 2^{b_1} 3^{a_1} 5^{c_1} \cdot \ldots \) and \( 2^{b_2} 3^{a_2} 5^{c_2} \cdot \ldots \).

i. Continuing with Part h, if \( f(q_1) = f(q_2) \), then what do you know about \( a_1, b_1, c_1, \ldots \) and \( a_2, b_2, c_2, \ldots \)? \( a_1 = a_2, b_1 = b_2, c_1 = c_2, \ldots \)

j. What do you know about \( q_1 \) and \( q_2 \)? Why? They are equal, because they have the same prime factorization.

k. Is \( f \) one-to-one? Yes. You just showed that \( f(q_1) = f(q_2) \) forces \( q_1 = q_2 \).

l. To show that \( f \) preserves the operation, we would need to show that \( f(q_1 \cdot q_2) \) is equal to what? \( f(q_1) \cdot f(q_2) \)

m. Suppose \( q_1 = 2^{a_1} 3^{b_1} 5^{c_1} \cdot \ldots \) and \( q_2 = 2^{a_2} 3^{b_2} 5^{c_2} \cdot \ldots \). What is \( q_1 \cdot q_2 \)? \( 2^{a_1 + a_2} 3^{b_1 + b_2} 5^{c_1 + c_2} \cdot \ldots \)

n. What is \( f \) of the thing in Part m? \( 2^{b_1 + b_2} 3^{a_1 + a_2} 5^{c_1 + c_2} \cdot \ldots \)

o. From part h, what is \( f(q_1) \cdot f(q_2) \)? \( 2^{b_1} 3^{a_1} 5^{c_1} \cdot \ldots \) \( \cdot 2^{b_2} 3^{a_2} 5^{c_2} \cdot \ldots \) \( = 2^{b_1 + b_2} 3^{a_1 + a_2} 5^{c_1 + c_2} \cdot \ldots \)

p. Does \( f \) preserve the operation? Yes. You just showed this.

q. What other thing must we prove to show that \( f \) is an isomorphism? The other main thing is to establish that \( f \) is onto.
2. For each of the following, give an example of a function \( f : \mathbb{R} \to \mathbb{R} \) that satisfies the given condition, and also give an example of one that does not. If no example exists, say so. You may draw graphs to define your functions.

a. \( f(5) = f(10) \). \( f(x) = (x - 5)(x - 10) = x^2 - 30x + 50 \) satisfies. \( f(x) = x \) does not.

b. \( f(5) = f(5) \). Every function satisfies this condition.

c. \( f(x) = f(x) \) for all \( x \in \mathbb{R} \). Every function satisfies this condition.

d. Whenever \( f(x) = f(y) \), then \( x = y \). This is the definition for one-to-one. \( f(x) = e^x \) satisfies. \( f(x) = x^2 \) does not.

e. Whenever \( y \) is an element of \( R \), there is an element \( x \) such that \( f(x) = y \). This is the definition for onto. \( f(x) = x \) satisfies. \( f(x) = x^2 \) does not.

f. \( f(x) = 5 \) for some \( x \) in \( \mathbb{R} \). You must hit 5 as a function value. \( f(x) = x^2 \) satisfies, but \( f(x) = -x^2 \) does not.

g. \( f(x) \neq 5 \) for every \( x \in \mathbb{R} \). This is the opposite of Part f. Just reverse your examples.

3. Draw the graph of a function \( f : \mathbb{R} \to \mathbb{R} \) that satisfies the following.

a. \( f \) is one-to-one and onto. The graph of \( f(x) = x \) works.

b. \( f \) is one-to-one, but is not onto. The graph of \( f(x) = e^x \) works.

c. \( f \) is onto, but is not one-to-one. The graph of \( f(x) = (x + 1)x(x - 1) = x^3 - x \) works.

d. \( f \) is neither one-to-one nor onto. The graph of \( f(x) = x^2 \) works.

4. The dihedral group \( D_4 \) has two four-element subgroups (and maybe a few more), \( R = \{ 1, 90^\circ, 180^\circ, 270^\circ \} \) and \( F = \{ 1, |, 180^\circ, \ldots \} \).

a. Give a substantive reason for believing that \( R \) and \( F \) are not isomorphic. \( 90^\circ \) and \( 270^\circ \) have order 4, and all of the elements of \( F \) have order 2, so we would have a lot of trouble finding an isomorphism between \( R \) and \( F \).

b. \( R \) or \( F \) is isomorphic to the group \( G = \{ e, a, b, c \} \) with the following multiplication table. Explicitly define an isomorphism \( f : G \to R \) or \( f : G \to F \). \( f : e \mapsto 1, a \mapsto 90^\circ, b \mapsto 180^\circ, c \mapsto 270^\circ \)

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Table 1. The multiplication table for \( G \).
5. Consider the following list of real numbers from the open interval \((0, 1)\).

\[
\begin{align*}
x_1 &= 0.a_{11}a_{12}a_{13}a_{14} \cdots \\
x_2 &= 0.a_{21}a_{22}a_{23}a_{24} \cdots \\
x_3 &= 0.a_{31}a_{32}a_{33}a_{34} \cdots \\
&\vdots
\end{align*}
\]

where each \(a_{ij}\) is a digit from the decimal expansion of \(x_i\). The number \(y = 0.b_1b_2b_3 \cdots\) so that each \(b_i\) is a digit from \{1, 2, 3, 4, 5, 6, 7, 8\} such that \(b_i \neq a_{ii}\).

a. Is it possible that \(b_2 = a_{22}\)? No, because they have different second digits.

b. Is it possible that \(b_3 = a_{23}\)? Yes. This is OK.

c. Is it possible that \(y = x_1\)? No, because they differ in the seventh digit.

d. Is it possible that \(y = x_i\) for some \(i\)? No, because they differ in the \(i\)-th digit.

e. Is it possible that this list contains all of the real numbers in \((0, 1)\)? No, because \(y\) is not on the list.

f. Is it possible that some other list contains all of the real numbers in \((0, 1)\)? No, because this list could just as easily be that other list.

g. Is it possible that this list contains all of the rationals in \((0, 1)\)? Yes. \(\mathbb{Q}\) is countable, so it’s possible to all of the rationals into a list like this. The number \(y\) will end up being irrational in this case.

h. Is it possible that every number in this list is the same number? Yes. That’s allowable.

i. Can every list of real numbers from \((0, 1)\) be written in this form? Yes. There is nothing assumed about this list other than that the numbers come from \((0, 1)\).

6. In the figure below is a knot \(K\). I want you to \textit{factor} it. That is, I want you to draw two knots \(A\) and \(B\) such that \(A\#B = K\).
7. Four generators for the trefoil group are shown below in Figure 1.

![Figure 1](image1.png)

**Figure 1.** Four loops passing through the trefoil.

a. Draw in the loop $acb^{-1}c$.

![Figure 2](image2.png)

**Figure 2.** Draw in the loop $acb^{-1}c$

b. The trefoil group has relations

\[
\begin{align*}
\text{b} * \text{a} &= \text{d} \\
\text{c} * \text{b} &= \text{d} \\
\text{a} * \text{c} &= \text{d}
\end{align*}
\]

Solve the second and third equations for $b$ and $a$. Express $acb^{-1}c$ in terms of only $c$ and $d$. $b = c^{-1}d$ and $a = dc^{-1}$. Be careful. This group is not abelian. Then $acb^{-1}c = (dc^{-1})c(d^{-1}c)c = c^2$. (corrected)

c. Draw the loop you found in Part b. Drawing $c^2$ is easy, so I’ll let you figure that out.
Figure 4. Draw the loop from Part b.