End Behavior for linear and Quadratic Functions

A linear function like \( f(x) = 2x - 3 \) or a quadratic function \( f(x) = x^2 + 5x + 3 \) are pretty generic. We have the tools to determine what the graphs look like just by looking at the functions. Today, I want to start looking at more general aspects of these functions that carry through to the more complicated polynomial functions (e.g., \( f(x) = 2x^3 - 3x^3 + 7x^2 - x + 11 \)). Specifically, I want to look at what the graphs look like on the ends, what they look like near the \( x \)-axis, and distinguishing aspects of the graph like bumps and wiggles.

Let’s start with end behavior. We’ve seen this so far as the ends of the curves we’ve drawn that point up or down and signify that the functions run off to positive or negative infinity (\( \pm \infty \)). In particular, we want to describe how the functions behave for very large values of \( x \).

First, let’s look at the function \( f(x) = x^2 + 5x + 3 \) at a somewhat large number, like \( x = 1,000,000 \) for example. We get

\[
f(1,000,000) = (1,000,000)^2 + 5(1,000,000) + 3.
\]

If we look at each term separately, we get the numbers

\[
\begin{align*}
1,000,000,000,000 \\
5,000,000 \\
3
\end{align*}
\]

Imagine graphing the point \((1,000,000, 1,000,005,000,003)\) (Good luck!). It wouldn’t look much different from \((1,000,000, 1,000,000,000,000)\).

We will say that \( x^2 \) dominates \( x \), when \( x \) is very large. Similarly, \( x \) dominates any constant. Notice that the same thing happens for large negative numbers like \( x = -1,000,000 \). Out towards the ends of the graph, therefore, \( f(x) = x^2 + 5x + 3 \) doesn’t look a whole lot different from \( f(x) = x^2 \), and when we’re just sketching graphs, they don’t look different at all. Similarly, the function \( f(x) = 2x - 3 \) looks a lot like \( f(x) = 2x \) for large values of \( x \).

![Figure 1:](image)

As another example, consider the linear function \( f(x) = -3x + 11 \). Since the \( x \)-term dominates the constant term, the end behavior is the same as the function \( f(x) = -3x \). For large positive values of \( x \), \( f(x) \) is large and negative, so the graph will point down on the right. Similarly, the graph will point up on the left, as on the left of Figure 1.

On the other hand, if we have the function \( f(x) = x^2 + 5x + 3 \), this has the same end behavior as \( f(x) = x^2 \), so the ends will go up on both sides, as on the right side of Figure ???. I’ve given these a little curve upwards to indicate that \( x^2 \) gets bigger faster than \( x \) does.
22 Factorings of Quadratic Functions

I’m going to assume that you can factor quadratic expressions, at least in the simpler cases. I want to focus on what information we can draw from the factorings.

Given the quadratic function \( f(x) = x^2 - 4x + 3 \), we can factor it as follows.

\[
 f(x) = (x - 1)(x - 3).
\]

Since anything times zero is zero, we can see that if \( x = 1 \) or if \( x = 3 \), we get \( f(1) = 0 \) and \( f(3) = 0 \). These are the places where the graph crosses the \( x \)-axis, as can be seen in Figure 2.

![Figure 2](image)

For very large values of \( x \) (both positive and negative), the magnitude of \( x^2 \) is much larger than \( x \), so it will dominate to the right and left. The sign on the \( x^2 \)-term, therefore, determines whether the parabola will open upwards or downwards.

For example, consider the function

\[
 g(x) = -2x^2 + 8x - 6,
\]

which is the first function multiplied by \(-2\). It will open downwards. In fact, we can factor as follows.

\[
 g(x) = -2x^2 + 8x - 6 = -2(x^2 - 4x + 3) = -2(x - 1)(x - 3).
\]

The \( x \)-intercepts are the same, \( x = 1, 3 \), but now everything is multiplied by a negative number, and that turns things upside-down. Look at Figure 3. The 2 stretches everything vertically, so the graph also looks skinnier.

The basic factorings give us three possibilities. If we shift the function function \( f \) up one unit, we get the following.

\[
 h(x) = x^2 - 4x + 4 = (x - 2)(x - 2).
\]

Since both factors are the same, only \( x = 2 \) is an \( x \)-intercept. The \( x^2 \)-term is positive, so the parabola opens upwards. The graph must look as it does in Figure 4, therefore.
Figure 3:

Figure 4:
If we shift the function up any higher, it won’t intersect the $x$-axis at all. This corresponds to the fact that the function
\[ j(x) = x^2 - 4x + 5 \]
does not factor over the real numbers. In fact, if we try to solve the equation $x^2 - 4x + 5 = 0$ using the quadratic formula, we get
\[
x = \frac{-(−4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{-4}}{2}.
\]
We have to use imaginary numbers to find square roots of negative numbers. Anyway, the graph is shown in Figure 5.

![Graph of $j(x) = x^2 - 4x + 5$]

We can also multiply by constants to stretch and compress the graphs vertically, and multiplying by negative numbers makes the parabolas open downwards.

### 2.1 Quiz 02-B

Sketch the graphs of the following quadratic functions. For the answers, give the $x$-intercepts and whether the graph opens up or down.

1. $f(x) = (x + 3)(x - 1)$.
2. $f(x) = (x + 1)^2$.
3. $f(x) = x^2 + 1$. This does not factor over the reals, and the vertex is at $x = 0$.
4. $f(x) = -3(x + 3)(x - 1)$.

### 3 Homework 04

For each of the given functions, find the $x$-intercept(s) and the end behavior.

1. $f(x) = x - 4$.
2. $f(x) = (x + 4)(x - 2)$.
3. $f(x) = (x - 3)^2$.
4. $f(x) = x^2 + x + 1$. This does not factor.
5. $f(x) = 2(x - 3)(x - 5)$.
6. $f(x) = -2(x + 1)(x + 1)$.
7. $f(x) = -x^2 - x - 1$. Compare this to problem 4.