1 Shapes of Cubic Functions

A cubic function (a.k.a. a third-degree polynomial function) is one that can be written in the form

\[ f(x) = ax^3 + bx^2 + cx + d. \tag{1} \]

Quadratic functions only come in one basic shape, a parabola. The parabola can be stretched or compressed vertically (making it look skinnier or wider), and it can be flipped up-side-down, but the graph is still always a parabola.

A cubic function has a bit more variety in its shape. We can get a lot of information from the factorization of a cubic function. We get a fairly generic cubic shape when we have three distinct linear factors. For example, consider the function

\[ f(x) = x^3 - 3x^2 - x + 3 = (x + 1)(x - 1)(x - 3), \tag{2} \]

which I am not expecting you to be able to factor (I just wanted to show you where it came from). This function must cross the \( x \)-axis exactly three times, so the graph in Figure 1 shouldn’t be too surprising.

![Figure 1](image)

The valley on the right side of the graph looks a little bit like a parabola. Shifting the graph upwards could make the three \( x \)-intercepts into two or perhaps just one.
For example, if we had a function that factored like
\[ g(x) = (x + 1)(x - 2)^2, \]
the graph would look like the one in Figure 2.

![Figure 2](image1)

If a function factored into a linear factor and a quadratic factor that was not factorable like
\[ h(x) = (x + 1)(x^2 - 4x + 5), \]
then the graph would look like the one in Figure 3.

![Figure 3](image2)

One way of telling that a quadratic is not factorable is to complete the square. In this last example, we would get
\[ x^2 - 4x + 5 = x^2 - 4x + 4 + 5 - 4 = (x - 2)^2 + 1. \]
The squared part is going to be zero or bigger, and then we’re adding one. This must be positive, and we can see this in the graph.
1 SHAPES OF CUBIC FUNCTIONS

1.1 Quiz 05A

In each of the following, find a cubic function with the given properties.

1. Has $x$-intercepts $x = -2, 3, 7$, and has a graph like the one in Figure 4.

   (a) \( f(x) = (x - 2)(x - 3)(x - 7) \)  
   (b) \( f(x) = (x + 2)(x - 3)(x - 7) \)  
   (c) \( f(x) = (x - 2)(x + 3)(x + 7) \)  
   (d) \( f(x) = (x + 2)(x + 3)(x + 7) \)  
   (e) none of these

![Figure 4](image-url)

2. Has $x$-intercepts $x = -1, 1$, and has a graph that looks like the one in Figure 5.

   (a) \( f(x) = (x - 1)^2(x + 1) \)  
   (b) \( f(x) = (x + 1)(x - 1)^2 \)  
   (c) \( f(x) = (x - 1)(x + 1)^2 \)  
   (d) \( f(x) = (x + 1)^2(x - 1) \)  
   (e) none of these

![Figure 5](image-url)
So far, however, the basic shapes we’ve seen have been pretty similar. The graphs have gone to $-\infty$ on the left, to $+\infty$ on the right, and there is a hill and a valley in the middle.

The other basic shapes we’ll see occur when the three factors line up at the same $x$.

For example, if we get three linear factors that are exactly the same, the hill and the valley come together. This can be seen in the function

\[ j(x) = (x - 2)^3. \]  

(5)

Here the graph looks like the one in Figure 6.

![Figure 6](image)

The last basic shape comes when we have a quadratic factor that doesn’t factor, and it lines up with a linear factor. For example, the quadratic expression \( x^2 - 4x + 5 \) will try to give us a valley above \( x = 2 \) as in the function \( h \) above and Figure 3. If this lines up with the linear factor \( (x - 2) \), which gives us a root at \( x = 2 \), we get a graph that looks like the one in Figure 7.

\[ k(x) = (x - 2)(x^2 - 4x + 5). \]  

(6)

![Figure 7](image)

Notice the difference between the graphs in Figures 6 and 7. The graph in Figure 6 becomes completely horizontal at \( x = 2 \), while the graph in Figure 7 just does a little wiggle.

As before, we can stretch one of the graphs above by multiplying by a number bigger than 1. Multiplying by a number smaller than one will compress the graph, and a negative number will flip everything up-side-down.
1.2 Quiz 05B

Find a cubic function that satisfies the given properties.

1. Find a cubic function that has a root at \( x = -2 \) and looks like Figure 8.
   (a) \( f(x) = (x + 2)^3 \)  
   (b) \( f(x) = (x - 2)^3 \)  
   (c) \( f(x) = (x + 2)^2(x - 2) \)  
   (d) \( f(x) = (x - 2)^2(x + 2) \)  
   (e) none of these

![Figure 8](image)

2. Find a cubic function that has roots at \( x = 0 \) and looks like Figure 9
   (a) \( f(x) = x^2(x - 1) \)  
   (b) \( f(x) = x^3 \)  
   (c) \( f(x) = x(x^2 + 1) \)  
   (d) \( f(x) = x(x + 1)^2 \)  
   (e) none of these

![Figure 9](image)
3. Find a cubic function that has roots at $x = -1, 1, 3$, and has a graph that looks like Figure 10.

(a) $f(x) = (x+1)(x-1)(x-3)$  
(b) $f(x) = (x-1)(x+1)(x+3)$  
(c) $f(x) = (-1)(x+1)(x-1)(x-3)$  
(d) $f(x) = (-1)(x-1)(x+1)(x+3)$  
(e) none of these

![Figure 10](image)

2. **Homework 05**

Find a cubic function that satisfies the following conditions.

1. Has roots $x = -3, -2, 2$.

(a) $f(x) = (x-3)(x-2)(x+2)$  
(b) $f(x) = (x+3)(x+2)(x+2)$  
(c) $f(x) = (x-3)(x-2)(x-2)$  
(d) $f(x) = (x+3)(x+2)(x-2)$  
(e) none of these

2. Has roots $x = 0, 1, 2$.

(a) $f(x) = (x+1)(x+2)$  
(b) $f(x) = x(x-1)(x-2)$  
(c) $f(x) = x(x+1)(x+2)$  
(d) $f(x) = (x-1)(x-2)$  
(e) none of these

3. Has roots $x = -2, 2$, and looks like Figure 11.

(a) $f(x) = -(x+2)(x-2)^2$  
(b) $f(x) = (x+2)(x-2)^2$  
(c) $f(x) = -(x+2)^2(x-2)$  
(d) $f(x) = (x+2)^2(x-2)$  
(e) none of these

![Figure 11](image)
4. Has roots $x = -2, 2$, and looks like Figure 12.

(a) $f(x) = (x + 2)^2(x - 2)$  
(b) $f(x) = -(x + 2)(x - 2)^2$  
(c) $f(x) = (x + 2)(x - 2)^2$  
(d) $f(x) = -(x + 2)^2(x - 2)$  
(e) none of these

Figure 12:

5. Has roots $x = -2$, and looks like Figure 13.

(a) $f(x) = (x - 2)^3$  
(b) $f(x) = (x + 2)(x^2 + 4x + 5)$  
(c) $f(x) = (x - 2)(x^2 - 4x + 5)$  
(d) $f(x) = (x + 2)^3$  
(e) none of these

Figure 13: